MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 6

3.3-2

First, we show that $x_n > 1$ for all n by mathematical induction. Indeed, $x_1 > 1$ is clear. Now if $x_{n-1} > 1$, we will have

$$x_n = 2 - \frac{1}{x_{n-1}} > 2 - \frac{1}{1} = 1$$

Hence $x_n > 1$ holds. Thus, $x_n > 1$ for all n.

On the other hand, we have

$$x_n - x_{n+1} = x_n - 2 + \frac{1}{x_n} = \frac{1}{x_n}(x_n - 1)^2 > 0$$

since $x_n > 1$. Hence we get $x_{n+1} < x_n$, which shows (x_n) is decreasing. And moreover, we know (x_n) is bounded below by 1 and above by x_1. By Monotone Convergence Theorem, we know $a = \lim x_n$ exists and apply limit at both side, we get

$$a = 2 - 1/a$$

Hence, $a = 1$ and we get $\lim x_n = 1$.

3.3-6

Let $f(x) = a + x - x^2$ and p the positive root of $f(x)$. (I.e. $p = \frac{1+\sqrt{1+4a}}{2}$). So we know that the following:

$$\begin{aligned} f(x) &\geq 0, \quad 0 \leq x \leq p \\ f(x) &\leq 0, \quad x \geq p \end{aligned}$$

Hence, we will have

$$z_{n+1} - z_n = \sqrt{a + z_n} - z_n = \frac{a + z_n - z_n^2}{\sqrt{a + z_n} + z_n} = \frac{f(z_n)}{\sqrt{a + z_n} + z_n}$$

Case 1, $z_1 \leq p$. We will show that $z_n \leq p$ and $z_n \leq z_{n+1}$ by induction. Clearly it holds for $n = 1$ since $f(z_1) \geq 0$.

Assume it holds for $n = k$, i.e. we have $z_k \leq z_{k+1} \leq p$. First, we have $z_{k+2} = \frac{f(z_{k+1})}{\sqrt{a+z_{k+1}}+z_{k+1}} + z_{k+1} \geq z_{k+1}$ by $z_{k+1} \leq p$ and $f(z_{k+1}) \geq 0$.

On the other hand,

$$f(z_{k+2}) = a + z_{k+2} - z_{k+2}^2 = a + z_{k+2} - a - z_{k+1} \geq 0$$

which implies $z_{k+2} \leq p$ and finish our induction.

Hence, we find z_n is increasing and bounded by p and implies $z = \lim z_n$ exists. Take limit of $z_{n+1} = \sqrt{a + z_n}$ at both side will give $f(z) = 0$ which implies $\lim z_n = z = p$. (z cannot be the negative root since z_n are all non-negative at least).

Case 2, $z_1 \geq p$. Similar trick with above. We can use mathematical induction to show that $z_n \geq z_{n+1} \geq p$. At this time, you just need to change signs of every inequalities in Case 1 and get the reversed result.

Hence, we will find z_n is bounded below and it is non-increasing. So the limit will exists and take limit at both side, we also get

$$\lim z_n = p$$

3.3-10

First, we note that y_n is bounded by above. Indeed, we have

$$y_n \leq \frac{1}{n} + \cdots + \frac{1}{n} = n \times \frac{1}{n} = 1$$

Second, we find that

$$y_{n+1} - y_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$$

Hence, (y_n) is an increasing sequence bounded above. Hence the limit exists by Monotone Convergence Theorem.

3.3-12(c)

Since the limit only depends on the tail of the sequence, we know that

$$\lim(1 + \frac{1}{n+1})^{n+1} = \lim(1 + \frac{1}{n})^n = e$$

We also note that $\lim \left(1 + \frac{1}{n+1}\right) = 1$, so we can apply the quotient of limit to get

$$\lim \left(1 + \frac{1}{n+1}\right)^n = \lim \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} = \frac{\lim \left(1 + \frac{1}{n+1}\right)^{n+1}}{\lim \left(1 + \frac{1}{n+1}\right)} = \frac{e}{1} = e$$