MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 5

3.2-1

(b) Suppose that $a := \lim x_n$ exists. Take $\varepsilon = \frac{1}{3}$ so that there exists a natural number K satisfying

$$|a - x_n| < \frac{1}{3} \quad \forall n \geq K.$$

Notice that $\frac{1}{2} \leq \frac{n}{n+1} < 1, \forall n \in \mathbb{N}$. If n is an odd natural number with $n \geq K$ this gives $\left|a + \frac{n}{n+1}\right| < \frac{1}{3}$, implying $-2 < -\frac{n}{n+1} - \frac{1}{3} < a < -\frac{n}{n+1} + \frac{1}{3} < 0$, i.e. $-2 < a < 0$. If n is an even natural number with $n \geq K$ this gives $\left|a - \frac{n}{n+1}\right| < \frac{1}{3}$, implying $0 < \frac{n}{n+1} - \frac{1}{3} < a < \frac{n}{n+1} + \frac{1}{3} < 2$, i.e. $0 < a < 2$. Since a cannot satisfy both inequalities simultaneously, a contradiction. Hence the sequence is divergent.
(c) We will show that $(\frac{n^2}{n+1})$ is divergence by contradiction. Assume $\lim \frac{n^2}{n+1} = a$ exists. Take $\epsilon = 1$. Note that

$$\frac{n^2}{n+1} > \frac{n^2 - 1}{n+1} = n - 1$$

So by A.P., we can choose a natural number K such that $K > a + 2$. In this case, we will have for any $n \geq K$,

$$\left|\frac{n^2}{n+1} - a\right| \geq \frac{n^2}{n+1} - a > n - 1 + a \geq K - 1 + a > 1$$

which implies $|x_n - a| < \epsilon = 1$ cannot hold for n large. This is a contradiction and hence the limit does not exists.

3.2-6(c)

We note that

$$\frac{\sqrt{n} - 1}{\sqrt{n} + 1} = 1 - \frac{2}{\sqrt{n} + 1}$$

We already know that

$$\lim \frac{1}{\sqrt{n}} = 0$$

by

$$\lim \frac{1}{n} = 0$$

and Theorem 3.2.10. Hence by Squeeze Theorem and $0 \leq \frac{1}{\sqrt{n}+1} \leq \frac{1}{\sqrt{n}}$, we have

$$\lim \frac{1}{\sqrt{n}+1} = 0$$

Hence

$$\lim \frac{\sqrt{n}-1}{\sqrt{n}+1} = 1 - \lim \frac{2}{\sqrt{n}+1} = 1 - 2 \lim \frac{1}{\sqrt{n}+1} = 1 - 2 \times 0 = 1$$

3.2-9

We only need to find the limit of this sequence. This will show the sequence $(\sqrt{n} y_n)$ converges.

Indeed, we note that

$$\sqrt{n} y_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

We can guess it will take limit $\frac{1}{2}$. Clearly, we will have

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \leq \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}} = \frac{1}{2}$$

On the other hand, we will have

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \geq \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}} = \frac{1}{2}\sqrt{1 - \frac{1}{n+1}}$$

Hence, we can apply the operation on these limits to get

$$\lim \frac{1}{n+1} = 0 \implies \lim 1 - \frac{1}{n+1} = 1 \implies \lim \sqrt{\frac{n}{n+1}} = 1 \implies \lim \frac{\sqrt{n}}{2\sqrt{n+1}} = \frac{1}{2}$$

Note that we already have

$$\frac{\sqrt{n}}{2\sqrt{n+1}} \leq \sqrt{n} y_n \leq \frac{1}{2}$$

Then by Squeeze Theorem, we get

$$\lim \sqrt{n} y_n = \frac{1}{2}$$

3.2-13

We note that

$$\begin{aligned}\sqrt{(n+a)(n+b)}-n &= \frac{(n+a)(n+b)-n^2}{\sqrt{(n+a)(n+b)}+n} = \frac{na+nb+ab}{\sqrt{(n+a)(n+b)}+n} \\ &= \frac{(a+b)+\frac{ab}{n}}{\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1}\end{aligned}$$

Since ab, a, b are all constant, we have

$$\lim \frac{ab}{n} = 0, \quad \lim \frac{a}{n} = 0 \quad \lim \frac{b}{n} = 0$$

Hence,

$$\begin{aligned}\lim (a+b) + \frac{ab}{n} &= a+b \\ \lim \sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1 &= \sqrt{\lim \left[(1+\frac{a}{n})(1+\frac{b}{n})\right]}+1 \\ &= \sqrt{(1+\lim \frac{a}{n})(1+\lim \frac{b}{n})}+1 = 2\end{aligned}$$

Hence

$$\lim \sqrt{(n+a)(n+b)}-n = \frac{\lim \left[(a+b)+\frac{ab}{n}\right]}{\lim \left[\sqrt{(1+\frac{a}{n})(1+\frac{b}{n})}+1\right]} = \frac{a+b}{2}$$

3.2-14(b)

From the inequalities $1 \le n!$ and $n! \le n^n$, we have $1 \le (n!)^{1/n^2} \le n^{1/n}, \forall n \in \mathbb{N}$. Since $\lim n^{1/n} = 1$, apply Theorem 3.2.7 and $\lim (n!)^{1/n^2} = 1$.

3.2-18

Let r be a number so that $1 < r < L$ and let $\varepsilon = L - r$. There exists a number $K \in \mathbb{N}$ so that if $n \ge K$ then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon$$

and

$$\frac{x_{n+1}}{x_n} > L - \varepsilon = r.$$

As $x_n > 0, \forall n \in \mathbb{N}$, $x_{n+K} > r x_{n+K-1} > \cdots > r^n x_K, \forall n \in \mathbb{N}$. Since $r > 1$, for any positive real number M, take n large enough satisfying $r^n > M/x_K$. Thus (x_n) is unbounded and divergent.