MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 2

2.2-5

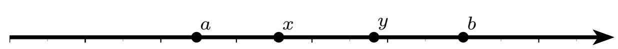
By Trichotomy Property, we have $x - y > 0$, $x - y = 0$ or $x - y < 0$.

First, if $x - y = 0$, then $|x - y| = 0 < b - a$.

Second, if $x - y > 0$, then we note $x < b, y > a$, we have $|x - y| = x - y < a - y < a - b$.

Similarly, if $x - y < 0$, then we note $x > a, y < b$, we have $|x - y| = y - x < a - x < a - b$.

For geometric meaning, we draw the picture as following.



As we can see, x, y will be located in the interval (a, b), then the distance of x, y will be strictly less than the length of interval (a, b).

2.2-10

(a) $|x - 1| > |x + 1| \iff (x-1)^2 > (x+1)^2 \iff x < 0$.

The solution set is $(-\infty, 0)$.

(b) If $x \le -1$, $|x| + |x+1| < 2 \iff -2x - 1 < 2 \iff x > -\frac{3}{2}$.

In this case, the solution set is $(-\frac{3}{2}, -1]$.

If $-1 < x < 0$, $|x| + |x+1| < 2 \iff 1 < 2 \iff -1 < x < 0$.

In this case, the solution set is $(-1, 0)$.

If $x \ge 0$, $|x| + |x+1| < 2 \iff 2x + 1 < 2 \iff x < \frac{1}{2}$.

In this case, the solution set is $[0, \frac{1}{2})$.

Combine all three cases and the solution set is $(-\frac{3}{2}, \frac{1}{2})$

2.3-7

S is bounded above since it has some upper bound which it contains. Denote the contained upper bound as u_0. From the definition of supremum, $\sup S \leq u_0$, since u_0 is an upper bound. On the other hand, that $u_0 \in S$ implies that $u_0 \leq \sup S$ because $s \leq \sup S, \forall s \in S$. Combine these two inequalities, $u_0 = \sup S$.

2.3-9

By definition of supremum, since $u - 1/n < \sup S$, $u - 1/n$ is not the upper bound of S. (Any number smaller than supremum will not become supremum.) An the other hand, since $u = \sup S$ is also a upper bound, and $u + 1/n > u$. So we get $u + 1/n$ is also a upper bound of S. (Any number bigger than an upper hand will still be an upper bound.)

2.3-12

Let's assume $a = \sup(S \cup \{u\}), b = \sup\{s^*, u\}$.

First, we will verify a is an upper bound of $\{s^*, u\}$. Since a is a supremum of $S \cup \{u\}$, it is an upper bound of $S \cup \{u\}$, and hence it is also an upper bound of S. Hence $a \geq \sup S = s^*$. Similarly, since a is an upper bound of $S \cup \{u\}$, we have $a \geq u$. From $a \geq s^*, a \geq u$, we get a is indeed an upper bound of $\{s^*, u\}$. So we get $a \geq \sup\{s^*, u\} = b$ by the definition of supremum.

Second, we will verify b is an upper bound of $S \cup \{u\}$. For any element $x \in S \cup \{u\}$, we have $x \in S$ or $x = u$. If we have $x \in S$, we will get $x \leq \sup S = s^* \leq b$ since supremum is always an upper bound. On the other hand, if $x = u$, then $x \leq \sup\{s^*, u\} = b$. So we get b is indeed an upper bound of $S \cup \{u\}$. Hence we have

$$b \geq \sup(S \cup \{u\}) = a$$

In summary, combining the previous result, i.e. $a \geq b$ and $b \geq a$, we get $\sup(S \cup \{u\}) = \sup\{s^*, u\}$