MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 1

Exercise 1

We will use the Algebraic Properties of $\mathbb{R}$ to proof.
a) $b \xlongequal{(A3)} 0 + b \xlongequal{(A4)} (-a) + a + b \xlongequal{(A2)} (-a) + (a + b) = (-a) + 0 \xlongequal{(A3)} -a$
b) $-(-a) \xlongequal{(A3)(A4)} a + (-a) + (-(-a)) \xlongequal{(A2)} a + ((-a) + (-(-a))) \xlongequal{(A4)} a + 0 \xlongequal{(A3)} a$
c) $(-1)a \xlongequal{(A3)(A4)} (-a) + a + (-1)a \xlongequal{(A2)(M3)} (-a) + (1a + (-1)a) \xlongequal{(D)(A4)} (-a) + 0a \xlongequal{\text{Theorem 2.1.2}(c),(A3)} -a$
d) $(-1)(-1) \xlongequal{(A3)(A4)} 1+(-1)+(-1)(-1) \xlongequal{(A2)(M3)} 1+(1(-1)+(-1)(-1)) \xlongequal{(D)(A4)} 1 + 0(-1) \xlongequal{\text{Theorem 2.1.2}(c),(A3)} 1$

Exercise 6

Otherwise, suppose that $(p/q)^2 = 6$ so that p, q are integers and no common integer factors other than 1. Since $p^2 = 6q^2$, p is even and q is odd. Denote $p = 2m$ and $q = 2n + 1$ for some integers m and n. Since $(2m)^2 = 6 \cdot (2n + 1)^2$, $6 = 4(m^2 - 6n^2 - 6n)$. This implies that 6 is a multiple of 4. Contradiction.

Exercise 11

a) Clearly, we will have $1/a \neq 0$. If not, we have $1 = a \cdot (1/a) = a \cdot 0 = 0$, a contradiction. By Theorem 2.1.8(a), we know $(1/a)^2 > 0$. Hence $a \cdot (1/a)^2 > 0$, which implies $1/a > 0$.

By Algebraic Properties of $\mathbb{R}$, we have $1/(1/a) \xlongequal{(M3)(M4)} (a\cdot(1/a))\cdot(1/(1/a)) \xlongequal{(M2)} a \cdot ((1/a) \cdot (1/(1/a))) \xlongequal{(M4)(M3)} a$
b) Since $1/2 > 0$, by Theorem 2.1.7 (c),

$$\frac{1}{2}a < \frac{1}{2}b$$

Again, use Theorem 2.1.7 (b), we have

$$\frac{1}{2}a + \frac{1}{2}a < \frac{1}{2}b + \frac{1}{2}a \text{ and } \frac{1}{2}a + \frac{1}{2}b < \frac{1}{2}b + \frac{1}{2}b$$

Using (D) algebraic properties of $\mathbb{R}$, and $(1/2) + (1/2) = 1 \cdot (1/2) + 1 \cdot (1/2) = 2 \cdot (1/2) = 1$, we have

$$a < \frac{1}{2}(a+b) \text{ and } \frac{1}{2}(a+b) < b$$

This is exactly what we want.

Exercise 22

We will show these results by Mathematical Induction.
a) Clearly, for $n = 1$, we have $c^1 \geq c$, and for $n = 2$, from $c > 1$, we have $c \cdot c > 1 \cdot c$, which is $c^2 > c$. Now assume for some $n = k \in \mathbb{N}$ with $k > 1$, we have $c^k > c$, then multiply both side by c, we get $c^{k+1} > c^2 > c$, hence the inequality holds for $n = k + 1$. By Mathemacial Induction, we know $c^n > c$ holds for all $n\mathbb{N}$ with $n > 1$. Clearly we have $c^n \geq c$.
b) Just as above, for $n = 1$, the statement $c^1 \leq c$ is clear. And for $n = 2$, $c < 1 \implies c^2 < c$. Still assume for $n = k$, $c^k \leq c$ holds for some $k > 1$, then we will have

$$c^{k+1} < c^2 < c$$

hence the statement $c^n < c$ and $c^n \leq n$ holds for all $n > 1$ with $n \in \mathbb{N}$.