MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 12

5.4-2

(i) Note that $\forall x, u \in A$, $0 < \frac{1}{x}, \frac{1}{u} \leq 1$. Given $\varepsilon > 0$, set $\delta = \varepsilon/2$. For any $x, u \in A$ satisfying $|x-u| < \delta$,

$$|f(x) - f(u)| = \left|\frac{x^2 - u^2}{x^2u^2}\right| = \left|\frac{1}{xu}\right| \left|\frac{1}{x} + \frac{1}{u}\right| |x-u| \leq 2\,|x-u| < 2\delta = \varepsilon.$$

Thus f is uniformly continuous on A.

(ii) Set $\varepsilon_0 = 1$, $(x_n) = (1/n)$ and $(u_n) = (1/(n+1))$. Note that $(x_n), (u_n) \subseteq B$ and $\lim(x_n - u_n) = 0$. Besides,

$$|f(x_n) - f(u_n)| = \left|n^2 - (n+1)^2\right| = 2n + 1 \geq 1.$$

By Criteria 5.4.2(iii), f is not uniformly continuous on B.

5.4-7

First, let's show f, g are all uniformly continous. For any $\epsilon > 0$, we note

$$|f(x) - f(u)| = |x - u|$$

and

$$|g(x) - g(u)| = |\sin x - \sin u| = 2|\sin\frac{x-u}{2}\cos\frac{x+u}{2}| \leq 2\left|\frac{x-u}{2}\right| = |x-u|$$

So if we choose $\delta = \epsilon$, then for any $|x-u| < \delta$, we have

$$|f(x) - f(u)| < \epsilon, |g(x) - g(u)| < \epsilon$$

Hence, f, g are all uniformly continuous.

But fg is not uniformly continous. Since we can choose $x_n = 2\pi n, u_n = 2\pi n + \frac{1}{n}$. Clearly, we have $\lim(x_n - u_n) = \lim\frac{1}{n} = 0$, and

$$|f(x_n)g(x_n) - f(u_n)g(u_n)| = \left|0 - (2\pi n + \frac{1}{n})\sin\frac{1}{n}\right| \geq 2\pi n \sin\frac{1}{n}$$

We note that from the important limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$, we have

$$\lim 2\pi n \sin\frac{1}{n} = 2\pi$$

So for n large enough, let's say $n > N$ for some integer N we will have $2\pi n \sin\frac{1}{n} \geq \pi$. Hence for $n > N$, we have

$$|f(x_n)g(x_n) - f(u_n)g(u_n)| \geq \pi$$

Hence by Nonuniform Continuity Criteria, we know fg is not uniformly continous.

5.4-8

Since f is uniformly continous, for any $\epsilon > 0$, we can find $\delta(\epsilon) > 0$, such that for any $|x - u| < \delta$, we have

$$|f(x) - f(u)| < \epsilon$$

Now since g is uniformly continous, we can find $\delta'(\delta) > 0$, such that for any $|y - v| < \delta'$, we have

$$|g(y) - g(v)| < \delta$$

Just put $x = g(y), u = g(v)$, since $|g(y) - g(v)| < \delta$, we will get

$$|f(g(y)) - f(g(v))| < \epsilon$$

Hence, we get $f \circ g$ is uniformly continuous.

5.4-10

We proof this result by contradiction. Assume f is not bounded. So at least, we can choose a sequence (x_n) with $|f(x_n)| > |f(x_{n-1})| + 1$.($x_1$ can be arbitrary. We can choose this sequence by induction.)

Now since (x_n) is bounded, we can choose a subsequence such that (x_{n_k}) is convergence. We denote it as (y_k). We still have $|f(y_k)| > |f(y_{k-1})| + n_k - n_{k-1} \geq |f(y_{k-1})| + 1$. Now we can choose $u_k = y_k$, $v_k = y_{k-1}$. So we have $\lim u_k - v_k = \lim y_k - y_{k-1} = 0$ but at the same time

$$|f(u_k) - f(v_k)| = |f(y_k) - f(y_{k-1})| \geq |f(y_k)| - |f(y_{k-1})| > 1$$

So f is not a uniformly continuous function by Nonuniform Continuity Criteria. This is a contradiction and hence, we get f is bounded on A