MATH 2050C Mathematical Analysis I
2019-20 Term 2

Solution to Problem Set 11

5.3-1

(**M1**)By the Maximum-Minimum Theorem 5.3.4, we have $\inf f(I) = f(x_0) > 0$ for some $x_0 \in I$. Denote $\alpha := \inf f(I)$. $f(x) \geq \inf f(I) = \alpha, \forall x \in I$.
(**M2**)The negation is that for any positive number β, there exist $x_\beta \in I$ so that $f(x_\beta) < \beta$. On the contrary, suppose the negation is true and that there exist $x_n \in [a,b]$ satisfying $0 < f(x_n) < 1/n$ for any $n \in \mathbb{N}$. Since $(x_n) \subseteq [a,b]$, there exists a subsequence (x_{n_k}) converges to some $x_0 \in \mathbb{R}$ by Theorem 3.4.8. We further have $x_0 \in [a,b]$ by the closedness of $[a,b]$. As f is continuous and $f(x_{n_k}) < 1/n_k, \forall k \in \mathbb{N}$,

$$f(x_0) = \lim f(x_{n_k}) \leq \lim 1/n_k = 0.$$

This contradicts against the condition $f(x) > 0, \forall x \in [a,b]$.

5.3-3

By the Maximum-Minimum Theorem 5.3.4, we can find $x_1, x_2 \in [a,b]$ with $f(x_1) = \max f(I), f(x_2) = \min f(I)$. We claim $f(x_1) \geq 0$ and $f(x_2) \leq 0$.

Now if we suppose $f(x_1) < 0$, then since $f(x_1)$ is a supremum of $f(I)$, for any $x \in [a,b]$, we have $f(x) \leq f(x_1)$. By the statement of problem, we can find $y \in [a,b]$ with $|f(y)| \leq \frac{1}{2}|f(x_1)|$. At least, we will have

$$-f(y) \leq \frac{1}{2}(-f(x_1)) \implies \frac{1}{2}f(x_1) \leq f(y) \implies f(x_1) < \frac{1}{2}f(x_1) \leq f(y)$$

which contridicts with the facts that $f(x_1)$ is an upper bound of $f(I)$.

Similarly, if we suppose $f(x_2) > 0$, we can find y with $\frac{1}{2}f(x_2) \geq f(y)$, which will implies $f(x_2) > f(y)$, which leads a contridiction with the fact $f(x_2)$ is a lower bound for $f(I)$.

So in summary, we have $f(x_1) \geq 0, f(x_2) \leq 0$. Hence, by Intermediate Value Theorem, we can find $c \in [x_1, x_2]$ or $c \in [x_2, x_1]$(depending on which is bigger) such that $f(c) = 0$.

5.3-12

Recall that $\sup\{a,b\} = \max\{a,b\} = \frac{a+b+|a-b|}{2}$, then we have

$$x^2 + \cos x \text{ continuous } \implies \left|x^2 - \cos x\right| \text{ continous } \implies f(x)\text{continuous}$$

And then by Maximum-Minimum Theorem, we can find an absolute minimum point $x_0 \in I$ for f on I.

Now we try to proof $\cos x_0 = x_0^2$. For any $x < x_0$, we know that $x^2 < x_0^2$. But we note $\max\{x^2, \cos x\} = f(x) \geq f(x_0) = \sup\{x_0^2, \cos x_0\} \geq x_0^2$. So we will have $x^2 > x_0^2$ or $\cos x > x_0^2$. But $x^2 > x_0^2$ is impossible by our choice of x, hence we can only have $\cos x > x_0^2$ for any $x < x_0$. Take a sequence (y_n) with $y_n \to x_0$ and $y_n < x_0$, we will have

$$\lim \cos y_n \geq x_0^2$$

and by the continuity of $\cos x$, we have $\cos x_0 \geq x_0^2$.

On the other direction, we need use the fact that $\cos x$ is a decreasing function on $[0, \frac{\pi}{2}]$. The proof is similar with above. For any $x > x_0$, we know that $\cos x_0 > \cos x$, and since $\max\{x^2, \cos x\} \geq \max\{x_0^2, \cos x_0\} \geq \cos x_0 > \cos x$, which imply $x^2 > \cos x_0$ for any $x > x_0$. Taking $x \to x_0$ from above and we have $x_0^2 \geq \cos x_0$.

Hence, combining the above results, we get x_0 is indeed a solution to the equation $\cos x = x^2$.

5.3-17

We proof the following claim first.

If $f : [0,1] \to \mathbb{R}$ is continous and has two different values a, b with $a < b$, then f cannot has only rational (or irrational) values.

Indeed, suppose $f(x_1) = a, f(x_2) = b$, then for any $c \in (a,b)$, by Intermediate Value Theorem, we can find $x_0 \in [x_1, x_2] or [x_2, x_1]$ such that $f(x_0) = c$. By Density Theorem, we can always find a rational number in (a,b), and we can also find a irrational number in (a,b). So this means f has to take rational values and irrational values. So the claim is proofed.

So from this claim, we can see that if f is not a constant, f will take at least two different values and we can write these two vaules as a, b with $a < b$ and hence f has to take both rational and irrational values.

Hence if f has only rational (irrational) values, f has to be a constant.