

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 7
New Due Date: 28th November 2019

Compulsory Part

1. Let m, n be relatively prime integers. Consider the map $\phi : \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ defined by:

$$\phi(a) = (a_m, a_n), \quad a \in \mathbb{Z}_{mn},$$

where a_l denotes the remainder of the division of a by l .

Show that ϕ is a ring isomorphism. (This is a version of the **Chinese Remainder Theorem**.)

2. Let R be a commutative ring. For $a \in R$, let:

$$I_a := \{x \in R \mid ax = 0\}$$

Show that I_a is an ideal of R .

3. Let I, J be ideals of a commutative ring R . Show that their intersection $I \cap J$ is an ideal of R .

Optional Part

1. Determine if each of the following maps is a ring homomorphism:

(a)

$$\begin{aligned} \phi : \mathbb{Z} &\longrightarrow \mathbb{Q}, \\ \phi(n) &= n^2, \quad n \in \mathbb{Z}. \end{aligned}$$

(b)

$$\begin{aligned} \phi : \mathbb{Z}_6 &\longrightarrow \mathbb{Z}_3, \\ \phi(s) &= s_3, \quad s \in \mathbb{Z}_6, \end{aligned}$$

where s_3 denotes the remainder of the division of s by 3.

(c)

$$\begin{aligned} \phi : \mathbb{Z}_7 &\longrightarrow \mathbb{Z}/3\mathbb{Z}, \\ \phi(s) &= s + 3\mathbb{Z}, \quad s \in \mathbb{Z}_7, \end{aligned}$$

where $s + 3\mathbb{Z}$ denotes the residue of s (viewed as an integer) in the quotient ring $\mathbb{Z}/3\mathbb{Z}$.

2. Find a ring homomorphism from \mathbb{Z}_7 to \mathbb{Z}_5 . If it does not exist, explain why not.
3. (a) Can there be a ring homomorphism from a non-integral domain to an integral domain? Why?

- (b) Can there be a ring homomorphism from an integral domain to a non-integral domain? Why?
4. Determine if the set I is an ideal in the ring R , where:
- $R = \mathbb{Z}[x]$, I is the set of polynomials $\sum_{i=0}^n a_i x^i$ in $\mathbb{Z}[x]$ with the property that a_0 is odd.
 - $R = \mathbb{Z}[x]$, I is the set of polynomials in $\mathbb{Z}[x]$ whose leading coefficients are even.
 - $R = \mathbb{Z}/6\mathbb{Z}$, I is the set of elements $r + 6\mathbb{Z} \in \mathbb{Z}/6\mathbb{Z}$ such that r is an even number. (Note that I is well-defined, since if one representative of $r + 6\mathbb{Z}$ is even, so is any other element in the same congruence class modulo $6\mathbb{Z}$.)
5. Let m, n be nonzero integers. Show that if $\gcd(m, n) = 1$, then $(mn) = (m) \cap (n)$ in the ring \mathbb{Z} .
6. An element a of a ring R is said to be a **nilpotent** element if $a^n = 0$ for some $n \in \mathbb{N}$. Suppose R is commutative. Show that the set N of nilpotent elements in R is an ideal.
7. Let R be a commutative ring. Let A and B be ideals in R . Define their sum and product, respectively, as follows:

$$A + B := \{r \in R : r = a + b \text{ for some } a \in A, b \in B.\}$$

$$AB := \left\{ r \in R : r = \sum_{i=1}^n a_i b_i \text{ for some } n \in \mathbb{N}, a_i \in A, b_i \in B. \right\}.$$

Show that the sets $A + B$ and AB are ideals in R .

8. Let R be the set of 2×2 real matrices of the form:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

Given that R is a commutative ring under the usual operations of addition and multiplication for matrices, show that R is isomorphic to the field \mathbb{C} .