# THE CHINESE UNIVERSITY OF HONG KONG
## Department of Mathematics
## MATH 2070A Algebraic Structures 2019-20
## Homework 5
## Due Date: 17th October 2019

**Compulsory Part**

1. Let $G = \{1, 2, 4, 5, 7, 8\}$. Define a binary operation $*$ on $G$ as follows:

$$l * k = \overline{l \cdot k},$$

where $\cdot$ represents the multiplication of integers, and for any $n \in \mathbb{Z}$ the symbol $\overline{n}$ denotes the remainder of the division of $n$ by $9$. Given that $G = (G, *)$ is group. Show that $G$ is isomorphic to $\mathbb{Z}_6$.

2. Let $G, G'$ be isomorphic cyclic groups. Show that for any generator $g$ of $G$ (i.e. $G = \langle g \rangle$) and any group isomorphism $\phi : G \longrightarrow G'$, the element $\phi(g)$ is a generator of $G'$.

3. Let:

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

   (a) Show that $(G, *)$ is a group, where $*$ is matrix multiplication.

   (b) Show that $(G, *)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

4. Show that a group $G$ is abelian if and only if the map

$$\phi : G \longrightarrow G$$

$$\phi(g) = g^{-1}, \quad g \in G,$$

is a group homomorphism.

**Optional Part**

1. Let $G = \{1, 5, 7, 11, 13, 17, 19, 23\}$. Define a binary operation $*$ on $G$ as follows:

$$l * k = \overline{l \cdot k},$$

where $\cdot$ represents the multiplication of integers, and for any $n \in \mathbb{Z}$ the symbol $\overline{n}$ denotes the remainder of the division of $n$ by $24$.

   (a) Given that $G = (G, *)$ is group, show that $G$ is *not* isomorphic to $\mathbb{Z}_8$.

   (b) $G$ is isomorphic to one of the following groups. Make a guess which one.

      i. $S_2 \times \mathbb{Z}_4$.

      ii. $\mathbb{Z}_3 \times \mathbb{Z}_5$.

      iii. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

2. Let $\phi : G \longrightarrow G'$ be a bijective group homomorphism. Show that the inverse map $\phi^{-1} : G' \longrightarrow G$ is also a group homomorphism.

3. Show that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$.

4. Show that any non-abelian group of order 6 is isomorphic to $S_3$.

5. Let $n$ be a positive integer. Define $\phi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}_n, +_n)$ as follows:

$$\phi(k) = \overline{k}, \quad k \in \mathbb{Z},$$

   where $\overline{k}$ denotes the remainder of the division of $k$ by $n$.

   (a) Show that $\phi$ is a group homomorphism.

   (b) Find $\ker \phi$ and the index $[\mathbb{Z} : \ker \phi]$.

   (c) Find all group homomorphism(s) $\psi : \mathbb{Z}_n \longrightarrow \mathbb{Z}$, if any exists.

6. Find the total number of group isomorphisms:

   (a) from $U_5$ to $U_5$.

   (b) from $U_{12}$ to $\mathbb{Z}_{12}$.

7. Define $\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{C} \backslash \{0\}, \cdot)$ as follows:

$$\phi(x) = e^{ix} = \cos x + i \sin x, \quad x \in \mathbb{R}.$$

   (a) Show that $\phi$ is a group homomorphism.

   (b) Find $\ker \phi$ and $\operatorname{im} \phi$.

8. Define a relation $\cong$ on groups as follows:

$$G \cong G' \quad \text{if } G \text{ is isomorphic to } G',$$

   Show that $\cong$ is an equivalence relation.

9. Let $G$ be a group. An isomorphism $\sigma : G \to G$ from $G$ onto itself is called an **automorphism** of $G$. Show that the set $\text{Aut}(G)$ of automorphisms of $G$ forms a group under composition.

10. (a) Let $G$ be a group and $S \subset G$ be a generating set for $G$, i.e. we have $G = \langle S \rangle$. Let $\lambda : G \to G'$ and $\mu : G \to G'$ be two homomorphisms from $G$ into a group $G'$ such that $\lambda(s) = \mu(s)$ for any $s \in S$. Show that $\lambda = \mu$.

    (b) Use (a) to compute the order of $\text{Aut}(\mathbb{Z}_{15})$. (More generally, what is the order of the automorphism group of a cyclic group of order $n$?)