THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 10
Due Date: 5th December 2019

Compulsory Part

1. Determine if the following rings are fields. Justify your answers.
   (a) $\mathbb{Q}[x]/(x^{17} + 5x^2 - 10x + 45)$
   (b) $\mathbb{Z}[x]/(x^6 - 210x - 616)$. (Note: It is $\mathbb{Z}[x]$ instead of $\mathbb{Q}[x]$.)
   (c) $\mathbb{Q}[x]/(4x^3 - 6x - 1)$
   (d) $\mathbb{R}[x]/(x^{17} + 5x^2 - 10x + 45)$

2. (a) Let a be a rational number. Show that the quotient ring $\mathbb{Q}[x]/(x - a)$ is isomorphic to $\mathbb{Q}$ by explicitly defining an isomorphism:
   $$\psi : \mathbb{Q} \longrightarrow \mathbb{Q}[x]/(x - a).$$
   (b) Show that $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{R}[x]/(x^2 + 2)$ by explicitly defining an isomorphism.

Optional Part

1. Consider the subfield $F = \mathbb{Q}(\sqrt[3]{5})$ of $\mathbb{R}$. Express the multiplicative inverse of $2+\sqrt[3]{5} \in F$ in the form:
   $$a + b\gamma + c\gamma^2, \quad a, b, c \in \mathbb{Q}; \gamma = \sqrt[3]{5}.$$

2. Let $F = \mathbb{F}_3$. Let $p = x^3 - x^2 + 1 \in F[x]$.
   (a) Show that $K = F[x]/(p)$ is a field.
   (b) Express the multiplicative inverse of $x^2 + 1 + (p) \in K$ in the form:
   $$a + bx + cx^2 + (p), \quad a, b, c \in F.$$

3. Find an irreducible polynomial $p \in \mathbb{Q}[x]$ such that:
   (a) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(2 - \sqrt{2})$.
   (b) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}\left(\sqrt{1 + \sqrt{3}}\right)$.
   (c) $\mathbb{Q}[x]/(p) \cong \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

4. (a) Show that $x^2 - 5$ is irreducible in $\mathbb{Q}(\sqrt{2})[x]$.
   (b) Show that $\mathbb{Q}(5 + \sqrt{2}) = \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(2 + \sqrt{5}) = \mathbb{Q}(\sqrt{5})$ as subfields of $\mathbb{R}$.
   (c) Show that $\sqrt{5}$ does not lie in $\mathbb{Q}(\sqrt{2})$.

(d) Conclude that $5+\sqrt{2}$ and $2+\sqrt{5}$ cannot be roots of the same irreducible polynomial in $\mathbb{Q}[x]$.

5. Let F be a subfield of a field E, and γ an element in E. Show that $F(a+b\gamma) = F(\gamma)$ for all nonzero $a, b \in F$.

6. (a) Show that $\sqrt{5}$ does not lie in $\mathbb{Q}(\sqrt{2})$.

   (b) Conclude that $5+\sqrt{2}$ and $2+\sqrt{5}$ cannot be roots of the same irreducible polynomial in $\mathbb{Q}[x]$.

7. Let F be a subfield of a field E, and γ an element in E. Let p, q be irreducible polynomials in $F[x]$ such that γ is a root of both p and q. Show that $q = up$ for some nonzero element $u \in F$.