1^{st} isomorphism Th. Let $\varphi: G \to G'$ be a gp hom. and $H = \ker\varphi$

Then $\bar{\varphi}: G/H \to \varphi(G)$, $gH \mapsto \varphi(g)$

is an isomorphism.

Rmk. We have the factorization

$$G \twoheadrightarrow G/H \xrightarrow{\simeq} \varphi(G) \hookrightarrow G'$$

Pf. ① $\bar{\varphi}$ is well-defined

If $aH = a'H$, then $a' = ah$ for some h.

$$\varphi(a') = \varphi(ah) = \varphi(a)\varphi(h) = \varphi(a).$$

② $\bar{\varphi}$ is a gp hom

$$\varphi((aH)(bH)) = \varphi(abH) = \varphi(ab) = \varphi(a)\varphi(b)$$

$$= \varphi(aH)\varphi(bH)$$

③ $\bar{\varphi}$ is injective.

$$\ker\bar{\varphi} = \{aH \mid \varphi(a) = 1_{G'}\} = \{H\}$$

④ $\bar{\varphi}$ is surj. follows from definition □.

This result is very useful in establishing isomorphisms

Examples. ① $\mathbb{Z} \to \mathbb{Z}_n$, $k \mapsto k \bmod n$.

$\ker = n\mathbb{Z}$. So $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

② $\mathbb{R} \to U(1) = \{z \in \mathbb{C}^\times \mid |z| = 1\}$

$t \mapsto e^{2\pi i t}$.

$\ker = \mathbb{Z}$. So $\mathbb{R}/\mathbb{Z} \cong U(1)$

③ $\mathbb{C}^\times \to \mathbb{R}^\times$, $z \mapsto |z|$.

$\ker = U(1)$. image $= \mathbb{R}_{>0}$. So

$\mathbb{C}^\times / U(1) \cong \mathbb{R}_{>0}$.

④ $GL_n(F) \to F^\times$, $A \mapsto \det A$.

$\ker = SL_n(F)$. So $GL_n(F)/SL_n(F) \cong F^\times$.

Direct product

Def Let H, K be gps. Define $H \times K$ direct product.

$(h,k)\cdot(h',k') := (hh', kk')$

Prop. Let $G = H \times K$. Set $\bar{H} = \{(h,e) \mid h \in H\} \trianglelefteq G$. Then

$G/\bar{H} \cong K$.

Pf. $G = H \times K \to K$, $(h,k) \mapsto k$.

surj, $\ker = \bar{H}$. So $G/\bar{H} \cong K$.

In general $\prod_{i=1}^{n} G_i$ direct product.

If G_i abelian, then one may also write $\bigoplus_{i=1}^{n} G_i$ instead of $\prod_{i=1}^{n} G_i$

This is direct sum.

⚠ There is a big difference between direct sum and direct product when ∞-many factors involved.

Example. The klein 4-gp $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. abelian, not cyclic.

$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ cyclic

Prop. Let $G = \mathbb{Z}_m \times \mathbb{Z}_n$. Then the order of $(1,1) \in G$ has order $\operatorname{lcm}(m,n)$.

Pf. Let k be the order. Then k is the smallest positive integer s.t. $k(1,1) = (0,0)$, i.e. $k \cdot 1_m = 0$ in $\mathbb{Z}_m$ and $k \cdot 1_n = 0$ in $\mathbb{Z}_n$.

So $m \mid k$, $n \mid k$. Thus $k = \operatorname{lcm}(m,n)$ $\square$

Cor. $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic iff $(m,n)=1$

Pf. $\Leftarrow$ $(1,1)$ has order $\mathrm{lcm}(m,n) = mn$.

So it generates the whole gp.

$\Rightarrow$ $\forall (a,b) \in \mathbb{Z}_m \times \mathbb{Z}_n$. $\mathrm{lcm}(m,n)(a,b) = (0,0)$.

So cyclic $\Rightarrow$ $\mathrm{lcm}(m,n) = mn$. $\square$

In general.

Prop. Let $(a_1, \cdots, a_n) \in \prod_{i=1}^{n} G_i$. Suppose that $|a_i| = r_i$.

Then $|(a_1, \cdots, a_n)| = \mathrm{lcm}(r_1, \cdots, r_n)$.

Pf. Leaves as an exercise

Structure of f. gen abelian gps

Th (Fundamental Th of f.g. abelian gps)

Every f.g. abelian gp G is iso to a direct product of cyclic gps of the form $\underbrace{\mathbb{Z}^r}_{\text{free / infinite part}} \times \underbrace{\mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_n^{k_n}}}_{\text{torsion / finite part}}$

where p_i are primes (not nec. distinct) and $k_i \in \mathbb{Z}_{>0}$.

Pf. Later (as a special case of f.gen mod of PID)

Rmk. $r \in \mathbb{N}$ is called the rank of G.

$m = p_1^{k_1} \cdots p_n^{k_n}$ is the order of the torsion part of G.

The above theorem implies that

For $m = p_1^{l_1} \cdots p_t^{l_t}$ prime factorization with p_i distinct.

$$\{\text{finite abelian gp of order } m\}/\text{iso}$$

$$\Updownarrow$$

$$\{\text{partitions of } l_i \text{ for each } i\}$$

$$l_i = l_{i1} + \cdots + l_{ij_i} \longleftrightarrow \prod_{i=1}^{t} (\mathbb{Z}_{p_i^{l_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{l_{ij_i}}})$$

Example: All abelian gps (up to isomorphism) of order $60 = 2^2 \cdot 3 \cdot 5$ are

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\cong$ $\qquad\qquad$ $\cong$

$\mathbb{Z}_{60}$ $\qquad\qquad$ $\mathbb{Z}_2 \times \mathbb{Z}_{30}$

Another classification result.

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_k}.$$

where $d_1 \mid d_2 \mid \cdots \mid d_k$.

The correspondence between 2 formulations

$$\boxed{\begin{matrix}4\\3\\5\end{matrix}} \qquad\qquad \boxed{\begin{matrix}2\\3\\5\end{matrix}}\ \boxed{\begin{matrix}2\\ \\ \\ \end{matrix}}$$

$$60 \qquad\qquad 30 \quad 2.$$