Normal subgps

Motivation 1: left coset = right coset.

Def. A subgp H of G is normal iff $gH = Hg \ \forall g \in G$.

i.e. each left coset is a right coset.

Notation $\subseteq$ subset. $<$ subgp. $\triangleleft$ normal subgp

Rmk. TFAE

(1) $gH = Hg \ \forall g \in G$

(2) $gH \subseteq Hg \ \forall g \in G$

(3) $Hg \subseteq gH \ \forall g \in G$

(4) $gHg^{-1} = H \ \forall g \in G$

(5) $gHg^{-1} \subseteq H \ \forall g \in G$

(6) $H \subseteq gHg^{-1} \ \forall g \in G$

Motivation 2: Natural gp structure on the set of left cosets G/H.

right coset $H\backslash G$.

Condition $(*)$. $(aH)(bH) = abH$

⚠ The definition depends on the choice of representatives.

Need to check that it is well-defined.

Thm. The subgp H of G is normal iff

$(*)$ $(aH)(bH) = abH$ is well-defined.

Pf. $\Rightarrow$ If $aH = a'H$, $bH = b'H$, then

$a' = ah_1$, $b' = bh_2$. Thus

$a'b'H = ah_1bh_2H$.

We have $ah_1bh_2 = (ab)(b^{-1}h_1b)h_2$.

Since $H = b^{-1}Hb$, $b^{-1}h_1b \in H$. and $(b^{-1}h_1b)h_2 \in H$.

So $abH = a'b'H$.

$\Leftarrow$. Set $b = g$. Then

$\forall h_1$, $ah_1bH = abH$.

i.e. $b^{-1}h_1b \in H$. So H is normal. $\square$

Thm. Let $H \triangleleft G$. Then $(*)$ defines a natural gp structure on G/H

Rmk. Here $G/H = H\backslash G$.

Pf. Associativity on G/H follows from associativity on G

Identity H.

inverse: The inverse of aH is $a^{-1}H$.

(check that it is well defined)

Def. G/H is called the quotient subgp.

Examples. ① Every subgp of an abelian gp is normal.

② $A_n \triangleleft S_n$

③ $\langle a \rangle \triangleleft D_n$, where $a \in D_n$ is the rotation by $2\pi/n$

④ $Sl_n \triangleleft Gl_n$

⑤ $G = S_3$, $a = (12)$. Then $\langle a \rangle \ntriangleleft G$

Motivation 3. kernel of a homomorphism

Prop. Let $\phi: G \to G'$ be a gp hom Then

(1) $\phi(1_G) = I_{G'}$.

(2) $\phi(a^{-1}) = \phi(a)^{-1}$.

(3) $H < G$, $\Rightarrow \phi(H) < G'$

(4) $H' < G' \Rightarrow \phi^{-1}(H') < G$.

Def. $\ker \phi = \phi^{-1}(1_{G'}) = \{ g \in G \mid \phi(g) = I_{G'} \}$

Prop. ϕ is injective iff $\ker(\phi) = \{I_G\}$

Pf. $\Rightarrow$ trivial

$\Leftarrow$ If $\phi(a) = \phi(a')$, then

$I_{G'} = \phi(a)\phi(a')^{-1} = \phi(a)\phi(a'^{-1}) = \phi(aa'^{-1})$

So $aa'^{-1} = I_G$ and $a = a'$. Thus injective $\square$

Prop. $\ker(\phi) \trianglelefteq G$.

Pf. We already know that $\ker(\phi) < G$ is a sbgp.

Let $a \in G$, $b \in \ker\phi$, then

$$\phi(aba^{-1}) = \phi(a)\phi(b)\phi(a^{-1}) = \phi(a)\, 1_{G'}\, \phi(a)^{-1} = 1_{G'}$$

So $aba^{-1} \in \ker\phi$. $\square$

Prop. Let $H \trianglelefteq G$. Then $\pi : G \to G/H$, $g \mapsto gH$

is a gp hom with $\ker(\pi) = H$.

Pf.

$$\begin{array}{c} \pi(ab) = abH \\ \parallel \\ \pi(a)\pi(b) = (aH)(bH) \end{array}$$

So π is a gp hom.

By def, $\ker(\pi) = H$. $\square$

Thus normal subgp $=$ ker of gp hom.