THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 7 Solution
Due Date: 31st October 2019

Compulsory part

1. Let H be a Sylow p-subgroup of G. Because q divides $|G|$, we know that $H \neq G$. For each $g \in G$, the conjugate group gHg^{-1} is also a Sylow p-subgroup of G. Because G has only one Sylow p-subgroup, it forces that $gHg^{-1} = H$ for all $g \in G$, so that H is a proper normal subgroup of G, and G is thus not simple.

2. First of all, $N[P]$ is contained in $N[N[P]]$. It suffices to show that $N[N[P]] \subset N[P]$. It is clear that $P \subset N[P]$. Let $a \in N[N[P]]$. One gets

$$aPa^{-1} \subset aN[P]a^{-1} = N[P].$$

It follows that P and aPa^{-1} are both Sylow p-group in $N[P]$. Since any two Sylow p-subgroups are conjugate, there is $b \in N[P]$ such that

$$bPb^{-1} = aPa^{-1}$$

implying that $a^{-1}b \in N[P]$. So $a \in N[P]$ as $b \in N[P]$. Then we are done with $N[N[P]] \subset N[P]$.

3. The divisors of 35^3 that are not divisible by 5 are 1, 7, 49, and 343, which are congruent to 1, 2, 4, and 3 respectively $\pmod 5$. By the third Sylow Theorem, there is only one Sylow 5-subgroup of a group of order 35^3, and it is a normal subgroup by Question 1.

4. If $m = 1$, then it is trivial because any p-group with order p^r where $r > 1$ is solvable. Suppose $m > 1$. The divisors of $p^r m$ that are not divisible by p are 1 and m. Because $m < p$, only 1 is congruent to 1 modulo p. By the third Sylow Theorem, there is a unique Sylow p-subgroup of a group of order $p^r m$ where $m < p$, and this must be a proper normal subgroup by Question 1. Thus such a group cannot be simple.

5. Let G be a group of order $(5)(7)(47)$. By the Sylow Theorems, there are only three subgroups of G having orders 5, 7 and 47, say H, K and L accordingly. Note that $HK \simeq H \times K$ as $H \cap K = \{e\}$. Similarly $G \simeq HK \times L \simeq H \times K \times L$ which is abelian and cyclic as each of H, K and L is abelian and cyclic.

6. Let G be a group of order 30. By the first Sylow Theorem, there are subgroups of G having orders 3 and 5.

   We claim that there is a normal subgroups of G of order 3 or 5. By the third Sylow Theorem we have $n_5 = 1$ or 6 and $n_3 = 1$ or 10. If $n_5 = 1$ or $n_3 = 1$, then we are done. Suppose $n_5 = 6$ and $n_3 = 10$. The group G then have 25 elements of order 5 and 20 elements of order 3, which is absurd as $|G| = 30$.

   So now there is a normal subgroup of G of order 3 or 5. We then have two possibilities.

- There is a normal subgroups of G of order 3, say H. G/H is of order 10, so by the Sylow theorem, there is a subgroup of order 5 in G/H. By the correspondence theorem, there is a subgroup of order 15.
- There is a normal subgroups of G of order 5, say K. G/K is of order 10, so by the Sylow theorem, there is a subgroup of order 3 in G/K. By the correspondence theorem, there is a subgroup of order 15.

Or we let H and K be the normal subgroups of G of order 3 and 5 accordingly. We then have $H \cap K = \{e\}$ and $HK < G$. So by the isomorphism theorem $|HK| = |H||K| = 15$.