THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 6
Due Date: 24th October 2019

Compulsory part

1. Let X be a G-set. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of X.

2. Let X be a G-set and $Y \subset X$. Let $G_Y = \{g \in G : gy = y \text{ for all } y \in Y\}$. Show that G_y is a subgroup of G.

3. Let G be the additive group of real numbers. Let the action $\theta \in G$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counterclockwise about the origin through θ radians. Let P be a point other than the origin in the plane.

   (a) Show $\mathbb{R}^2$ is a G-set.

   (b) Describe geometrically the orbit containing P.

   (c) Find the group G_P.

4. Let X_i for $i \in I$ be G-set for the same group G, and suppose the sets X_i are not necessarily disjoint. Let $X_i' = \{(x, i) : x \in X_i\}$ for each $i \in I$. Then the sets X_i' are disjoint, and each can still be regarded as a G-set in an obvious way. (The elements of X_i have simply been tagged by i to distinguish them form the elements of X_j for $i \neq j$.) The G-set $\cup_{i \in I} X_i'$ is the disjoint union of the G-sets X_i. Show that every G-set is isomorphic to a disjoint union of left coset G-sets.

5. (a) Let X be a transitive G-set and let $x_0 \in X$ and $g_0 \in G$. If $H = G_{x_0}$, describe $K = G_{g_0 x_0}$ in terms of H and g_0.

   (b) Based on part (a), conjecture conditions on subgroups H and K of G such that the left coset G-sets of H and K are isomorphic.

   (c) Prove your conjecture in part (b).

6. Up to isomorphism, how many transitive S_3-sets X are there?