THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 5 Solution
Due Date: 17th October 2019

Compulsory part

1. We use definition to show that HN is a group.

Closure: Let $h_1n_1, h_2n_2 \in HN$ where $h_1, h_2 \in H$ and $n_1, n_2 \in N$. Because N is a normal subgroup, left cosets are right cosets so $Nh_2 = h_2N$; in particular, $n_1h_2 = h_2n_3$ for some $n_3 \in N$. Then

$$(h_1n_1)(h_2n_2) = (h_1h_2)(n_3n_2) \in HN,$$

so HN is closed under the group operation.

Identity: $e = ee \in HN$ because $e \in H$ and $e \in N$.

Inverses: Now $(h_1n_1)^{-1} = n^{-1}h^{-1} \in Nh^{-1}$ and $Nh^{-1} = h^{-1}N$ as N is normal. Thus $n^{-1}h^{-1} = h^{-1}n^4$ for some $n^4 \in N$, so $(h_1n_1)^{-1} \in HN$,

Clearly HN is the smallest subgroup of G containing both H and N, because any such subgroup must contain all the products hn for $h \in H$ and $n \in N$.

2. The fact that K is normal shows that $hkh^{-1} \in K$, so $hkh^{-1}k^{-1} \in K$. Similarly, the fact that H is normal shows that $h(kh^{-1}k^{-1}) \in H$. Thus $hkh^{-1}k^{-1} \in H \cap K$, so $hkh^{-1}k^{-1} = e$. It follows that $hk = kh$.

3. (a) Let $\gamma G \to G/H$ be the natural homomorphism of a group onto its factor group. Then $\gamma[K] = K/H = B$ is a normal subgroup of $A = G/H$ as K is normal in G. Similarly $\gamma[L] = L/H = C$ is a normal subgroup of A. It is clear that $B = K/H < C = L/H$ since $K < L$.

   (b) By the third isomorphism theorem, one has

$$(A/B)/(C/B) \cong A/C \cong (G/H)/(L/H) \cong G/L.$$

4. First note that $G = K \vee L = KL = LK$. By the second isomorphism theorem, one has

$$G/L = KL/L \cong K/(K \cap L) \cong K/(\{e\}) \cong K$$

and

$$G/K = LK/K \cong L/(K \cap L) \cong L/(\{e\}) \cong L.$$

5. We use induction to show that $|H_i| = s_1s_2 \cdots s_i$ for $i = 1, 2, \cdots n$.

   For $n = 1$, $s_1 = |H_1/H_0| = |H_1/\{e\}| = |H_1|$.

   Now suppose that $|H_k| = s_1s_2 \cdots s_k$ for $k < i$. Notice that $|H_i/H_{i-1}| = s_i$. By our induction assumption, $|H_{i-1}| = s_1s_2 \cdots s_{i-1}$. Thus $|H_i| = |H_{i-1}|s_i = s_1s_2 \cdots s_{i-1}s_i$. Our induction is complete and the desired assertion follows by taking $i = n$.

6. A composition series for G contains a finite number of subgroups of G. If G is infinite and abelian, and
$$\{e\} = H_0 < H_1 < H_2 < \cdots < H_n = G$$
is a subnormal series, the factor groups H_i/H_{i-1} cannot all be of finite order for $i = 1, 2, \cdots, n$, or $|G|$ would be finite by the above Exercise. Suppose $|H_k/H_{k-1}|$ is infinite. Now every infinite abelian group has a proper nontrivial subgroup and hence a normal subgroup. To see this we need only consider the cyclic subgroup $\langle a \rangle$ for some $a \neq e$ in the group. If $\langle a \rangle$ is finite, we are done as $\langle a \rangle$ is a proper normal subgroup of the group. If $\langle a \rangle$ is infinite cyclic, then $\langle a^2 \rangle$ is a proper (normal) subgroup of the group. Thus we see that H_k/H_{k-1}, as an infinite abelian group, has a proper nontrivial subgroup, so it is not simple and our series is not a composition series.

Optional part

1. It suffices to show that the direct product of two solvable groups is solvable. Let G and H be solvable groups. Let
$$\{e_G\} < G_1 < \cdots < G_n = G,$$
and
$$\{e_H\} < H_1 < \cdots < H_m = H$$
be composition series for G and H respectively. $G \times \{e_H\}$ is a normal subgroup of $G \times H$, and for two subgroups K and L of H, if K is a normal subgroup of L, then $G \times K$ is a normal subgroup of $G \times L$. Therefore
$$\{e_G\} \times \{e_H\} < G_1 \times \{e_H\} < \cdots < G \times \{e_H\} < G \times H_1 < \cdots < G \times H$$
is a subnormal series for $G \times H$. (In fact it is a composition series). Furthermore, the quotient groups are all quotient groups from either the composition series of G, or the composition series of H, so they are simple. Therefore, $G \times H$ is solvable.

2. We first show that $H_{i-1}N \triangleleft H_iN$. Let $h_{i-1}n_1 \in H_{i-1}N$ and $h_in_2 \in H_iN$ where $h_{i-1} \in H_{i-1}$, $h_i \in H_i$ and $n_1, n_2 \in N$. Using the fact that $H_{i-1} \triangleleft N$ and that $N \triangleleft G$, we obtain
$$\begin{aligned}(h_in_2)h_{i-1}n_1(h_in_2)^{-1} &= h_in_2h_{i-1}n_1n_2^{-1}h_i^{-1} \\ &= h_ih_{i-1}n_3n_1n_2^{-1}h_i^{-1} = h'_{i-1}h_in_4h_i^{-1} = h'_{i-1}n_5 \in H_{i-1}N\end{aligned}$$
where $n_3, n_4, n_5 \in N$ and $h'_{i-1} \in H_{i-1}$. Thus $H_{i-1}N$ is a normal subgroup of H_iN.

By the second and the third isomorphism theorem we have
$$(H_iN)/(H_{i-1}N) \simeq H_i/\left(H_i \cap (H_iN)\right) \simeq (H_i/H_{i-1})/\left(\left(H_i \cap (H_iN)\right)/H_{i-1}\right).$$
On the other hand, H_i/H_{i-1} is simple implies that $\left(H_i \cap (H_iN)\right)/H_{i-1}$ is either trivial or isomorphic to H_i/H_{i-1}, so $(H_iN)/(H_{i-1}N)$ is either trivial or isomorphic to H_i/H_{i-1}. Because $N = H_0N$ is itself simple, it follows at once that the distinct groups among the H_iN for $i = 0, 1, 2, \cdots, n$ form a composition series for G.