THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 5
Due Date: 17th October 2019

Compulsory part

1. Let N be a normal subgroup of G and let H be any subgroup of G. Let $HN = \{hn | h \in H, n \in N\}$. Show that HN is a subgroup of G, and is the smallest subgroup containing both N and H.

2. Show that if H and K are normal subgroups of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$.

3. Let H, K and L be normal subgroups of G with $H < K < L$. Let $A = G/H$, $B = K/H$, and $C = L/H$.

   (a) Show that B and C are normal subgroups of A, and $B < C$.

   (b) To what factor group of G is $(A/B)/(C/B)$ isomorphic?

4. Let K and L be normal subgroups of G with $K \vee L = G$, and $K \cap L = \{e\}$. Show that $G/K \simeq L$ and $G/L \simeq K$.

5. Show that if
$$H_0 = \{e\} < H_1 < H_2 < \cdots < H_n = G$$
is a subnormal (normal) series of G, and if H_{i+1}/H_i is of finite order s_{i+1}, then G is of finite order $s_1 s_2 \cdots s_n$.

6. Show that an infinite abelian group can have no composition series.

Optional part

1. Show that a finite direct product of solvable group is solvable.

2. Let $H_0 = \{e\} < H_1 < H_2 < \cdots < H_n = G$ be a composition series for a group G. Let N be a normal subgroup of G, and suppose that N is a simple group. Show that the distinct groups among H_0, H_iN for $i = 0, \cdots n$ also form a composition series for G. [Hint: H_iN is a group. Show that $H_{i-1}N \triangleleft H_iN$. By the second and the third isomorphism theorem we have
$$(H_iN)/(H_{i-1}N) \simeq H_i/\left(H_i \cap (H_iN)\right) \simeq (H_i/H_{i-1})/\left(\left(H_i \cap (H_iN)\right)/H_{i-1}\right).$$
But H_i/H_{i-1} is simple.]