THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 4 Solution
Due Date: 3rd October 2019

Compulsory part

1. Note that $|\phi[G]| < |G|$ from the definition $\phi[G] = \{\phi(g) : g \in G\}$. Hence $|\phi[G]|$ is finite. We have $|\phi[G]| = |G/Ker(\phi)|$, so $|\phi[G]|$ is a divisor of $|G|$.

2. We first note that $Ker(\phi)$ is a subgroup of G. By the Theorem of Lagrange, either $Ker(\phi) = \{e\}$ or $Ker(\phi) = G$ as $|G|$ is a prime number. If $Ker(\phi) = \{e\}$, then ϕ is one to one. If $Ker(\phi) = G$, then the map ϕ is the trivial homomorphism, mapping everything into the identity element.

3. Let $x', y' \in \phi[G]$ and let $\phi(x) = x'$ and $\phi(y) = y'$ where $x, y \in G$. Then $\phi[G]$ is abelian which is equivalent to

$$x'y' = y'x' = e' \Leftrightarrow x'^{-1}y'^{-1}x'y' = e' \Leftrightarrow \phi\left(x^{-1}y^{-1}xy\right) = e' \Leftrightarrow x^{-1}y^{-1}xy \in Ker(\phi).$$

4. The necessary and sufficient condition is $hk = kh$.

   If ϕ is a homomorphism, we then have

$$hk = \phi\left(1,0\right) + \phi\left(0,1\right) = \phi\left(1,1\right) = \phi\left(0,1\right) + \phi\left(1,0\right) = kh.$$

   Conversely, suppose that $hk = kh$. For any (i, j) and (m, n) in $\mathbb{Z} \times \mathbb{Z}$, one has

$$\phi\left((i,j) + (m,n)\right) = \phi\left(i+m, j+n\right) = h^{i+m}k^{j+n} = \phi(i,j)\phi(m,n).$$

5. $\{e^{q\pi i} : q \in \mathbb{Q}\}$

6. S_3

7. Suppose that $G/Z(G)$ is cyclic and is generated by the coset $aZ(G)$. Let $x, y \in G$. Then $x \in a^m Z(G)$ and $y \in a^n Z(G)$ for some integers m, n. We can thus write $x = a^m z_1$ and $y = a^n z_2$ where $z_1, z_2 \in Z(G)$. Because z_1 and z_2 commute with every element of G, we have

$$xy = a^m z_1 a^n z_2 = a^{m+n} z_1 z_2 = a^n z_2 a^m z_1 = yx,$$

   showing that G is abelian. Therefore by contrapositive, if G is not abelian, then $G/Z(G)$ is not cyclic.

8. Let G be a nonabelian group of order pq. Suppose its centre subgroup $Z(G)$ is not trivial. Note that $Z(G) \neq G$ as G is nonabelian. By the theorem of Lagrange, $|Z(G)|$ divides pq, so $|Z(G)|$ can only be either p or q and hence is cyclic. But both cases contradicts the preceding problem. We only have $Z(G) = \{e\}$.

Optional part

1. Let S_a be the set $\{x \in G : \phi(x) = \phi(a)\}$. Let $s \in S_a$. We then have $\phi(sa^{-1}) = \phi(a)\phi(a^{-1}) = e'$, implying that $sa^{-1} \in H$ or equivalently $s \in Ha$.

   Let $h' \in Ha$. Then $h' = ha$ for some $h \in H$. This h' is in S_a as

$$\phi(h') = \phi(ha) = \phi(h)\phi(a) = e'\phi(a) = \phi(a).$$

2. The preceding exercise shows that the map ϕ is a homomorphism for all choices of h and k in G if and only if $hk = kh$ for all h and k in G, that is, if and only if G is an abelian group.

3. (a) All 3-cycle are even.

   (b) Every element in A_n can be written as the product of an even number of 2-cycles. We then pair up the adjacent 2-cycles and thus it suffices to show that the product of any pair of 2-cycles can be written as the product of some 3-cycles. For distinct i, j, k, ℓ, we have the following three possibilities:

$$\begin{pmatrix} i & j \end{pmatrix} \begin{pmatrix} i & j \end{pmatrix} = Id,$$

$$\begin{pmatrix} i & j \end{pmatrix} \begin{pmatrix} j & k \end{pmatrix} = \begin{pmatrix} i & j & k \end{pmatrix},$$

   and

$$\begin{pmatrix} i & j \end{pmatrix} \begin{pmatrix} k & \ell \end{pmatrix} = \begin{pmatrix} i & j & k \end{pmatrix} \begin{pmatrix} j & k & \ell \end{pmatrix}.$$

   (c) By the hint, we find that

$$(r, s, i)^2 = (r, i, s),\ (r, s, j)(r, s, i)^2 = (r, i, j),\ (r, s, j)^2(r, s, i) = (s, i, j),$$

   and

$$(r, s, i)^2(r, s, k)(r, s, j)^2(r, s, i) = (i, j, k).$$

   Note that every 3-cycle either contains neither r nor s and is of the form (i, j, k), or just one of r or s and is of the form (r, i, j) or (s, i, j), or both r and s and is of the form (r, s, i) or $(r, i, s) = (s, r, i)$. Because all of these forms can be obtained from our special 3-cycles (see the above), we see that the special 3-cycles generate A_n.

   (d) Following the hint, we find that

$$((r, s)(i, j))\,(r, s, i)^2\,((r, s)(i, j))^{-1} = (r, s, j).$$

   If N is a normal subgroup of A_n and contains a 3-cycle, which we can consider to be (r, s, i) in which r and s could be any two fixed numbers from 1 to n as in Part(c), we see that N must contain all the special 3-cycles and hence be all of A_n by Part(c).

   (e) Before drilling into the computations in the hints of the five cases, we observe that one of the cases must hold. If Case I is not true and Case II is not true, then when elements of N are written as a product of disjoint cycles, no cycle of length greater than 3 occurs, and no element of N is a single 3-cycle. The remaining cases cover the possibilities that at least one of the products of disjoint cycles involves two cycles of length 3, involves one cycle of length 3, or involves no cycle of length 3. Thus all possibilities are covered, and we now turn to the computations in the hints.

   Case I If N contains a 3-cycle, then we are done by Part (d).

Case II N contains a product of disjoint cycles, at least one of which has length greater than 3. Suppose N contains the disjoint product $\sigma = \mu(a_1, a_2, \cdots, a_r)$. We then have that

$$\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} = (a_1, a_3, a_r)$$

which is in N as $\sigma^{-1} \in N$ and $(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} \in N$ by the normality. Thus in this case, N contains a 3-cycle and is equal to A_n by Part(d) again.

Case III With a similar reason in Case II, we see that

$$\sigma^{-1}(a_1, a_2, a_4)\sigma(a_1, a_2, a_4)^{-1} = (a_1, a_4, a_2, a_6, a_3) \in N.$$

Thus N contains a cycle of length greater than 3, and $N = A_n$ by using the method in Case II.

Case IV Since $\sigma \in N$, $\sigma^2 \in N$. On the other hand, by noting that μ is a product of disjoint 2-cycles, we have $\sigma^2 = (a_1, a_3, a_2)$, so N contains a 3-cycle. Hence $N = A_n$ by Part (d).

Case V With a similar reason in Case II, we see that

$$\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1} = (a_1, a_3)(a_2, a_4) \in N.$$

Letting $\alpha = (a_1, a_3)(a_2, a_4)$ and $\beta = (a_1, a_3, i)$ where i is different from a_1, a_2, a_3, a_4, we have $\alpha \in A_n$ and $\beta \in N$. N is a normal subgroup of A_n implies that $\beta^{-1}\alpha\beta\alpha \in N$. A direct computation yields that $\beta^{-1}\alpha\beta\alpha = (a_1, a_3, i)$. Thus $N = A_n$ in this case also, by Part(d).