THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Homework 3 Solution
Due Date: 26th September 2019

Compulsory part

1. For $n \geq 2$, then $|A_n|$ is of index 2, so A_n is a normal subgroup of S_n. And also S_n/A_n is isomorphic to $\mathbb{Z}_2$. If $n = 1$, then $A_n = S_n$ so S_n/A_n is the trivial group of only one element.

2. Note that the torsion group T is a subgroup of G (Why?). Because G is abelian, every subgroup of G is a normal subgroup, so T is normal in G. Suppose that xT is of finite order in G/T; in particular, we have $(xT)^m = T$ for some positive integer m. Then $x^m \in T$. Because T is a torsion group, we must have $(x^m)^r = x^{mr} = e$ in G for some positive integer r. Thus x is of finite order in G, so that $x \in T$. This means that $xT = T$. Thus the only element of finite order in G/T is the identity T, so G/T is a torsion free group.

3. We have $|G/H| = m$. Because the order of each element of a finite group divides the order of the group, we see that $(aH)^m = H$ for all elements $aH \in G/H$. On the other hand we have $a^m H = (aH)^m$, so we see that $a^m \in H$ for all $a \in G$.

4. Let H be the intersection of all subgroups of G that are of order s. Notice that H is still a subgroup of G. Note that if $K < G$ is of order s, then for any $g \in G$ $gKg^{-1} < G$ is also of order s. So $H = gHg^{-1}$ for any $g \in G$. Hence H is a normal subgroup.

5. Let $H = \{g \in G | i_g = i_e\}$. This H is a subgroup of G as for any $a, b \in H$, we have for any $x \in G$,
$$\left(ab^{-1}\right) x \left(ab^{-1}\right)^{-1} = ab^{-1}xba^{-1} = axa^{-1} = x.$$
Let $h \in H$. For any $g \in G$, $i_{ghg^{-1}} = i_e$ since for any $y \in G$
$$\left(ghg^{-1}\right) y \left(ghg^{-1}\right)^{-1} = ghg^{-1}ygh^{-1}g^{-1} = gg^{-1}ygg^{-1} = y.$$
This shows that $ghg^{-1} \in H$ for any $g \in G$.

6. (a) The $n \times n$ matrices with determinant 1 form a subgroup of $GL(n, \mathbb{R})$. The normality follows from $\det(AXA^{-1}) = \det(A)\det(X)\det(A^{-1}) = 1$ for any X having determinant 1.

   (b) The $n \times n$ matrices with determinant ± 1 form a subgroup of $GL(n, \mathbb{R})$. The normality follows from $\det(AXA^{-1})^2 = \det(A)^2 \det(X)^2 \det(A^{-1})^2 = 1$ for any X having determinant ± 1.

Optional Part

1. Let $\{H_i | i \in I\}$ be the set of all normal subgroups of G containing S. Note that G is such a subgroup of G, so the index set I is nonempty. Let $K = \bigcap_{i \in I} H_i$. By Problem 4, we know that K is a normal subgroup of G, and of course K contains S because every H_i contains S. From the constructions, we see that K is contained in every normal subgroup H_i of G containing S, so K must be the smallest normal subgroup of G containing S.

2. (a) The associativity is clear. The subset $\{e\}$ acts as identity for this multiplication. Note that $\emptyset$ has no inverse, so $P(G)$ is not a group under this operation.

   (b) Note that if N is a normal subgroup, then we have $(ab)N = (aN)(bN)$ for any $a, b \in G$.

   (c) Associativity is clear. The coset N acts as identity element. The equation

$$(a^{-1}N)(aN) = eN = (aN)(a^{-1}N)$$

   shows that each coset has an inverse, so these cosets of N do form a group under this multiplication. The identity of the coset group is N, while the identity for the multiplication of all subsets of G is $\{e\}$. If $N \neq \{e\}$, these identities are different.