THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 5
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1. Which of the following groups are solvable?

   (a) D_n

   (b) S_n

   (c) $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad \neq 0 \right\} < GL(2, \mathbb{R})$

Solution. (a)
- Recall that D_n is the disjoint union of two sets, one consisting of n rotations and the other consisting of n reflections.
- The rotations form a subgroup and is generated by the rotation σ by $\frac{2\pi}{n}$ rad.
- We show that

  $$\{\text{Id}\} < \langle\sigma\rangle < D_n$$

  is a solvable series.
  - $\langle\sigma\rangle/\{\text{Id}\} \simeq \langle\sigma\rangle$ which is abelian.
  - $D_n/\langle\sigma\rangle \simeq \mathbb{Z}_2$ which is also abelian. (Why $\langle\sigma\rangle \triangleleft D_n$?)
- It follows that D_n is solvable.

(b)
- We show that S_n is solvable if and only if $n \leq 4$.

$\mathbf{n \geq 5}$
- Suppose to the contrary that S_n is solvable.
- Then its subgroup A_n is solvable.
- Let $\{\text{Id}\} = G_0 < G_1 < G_2 < \cdots < G_{k-1} < G_k = A_n$ be a solvable series.
- Then G_{k-1} is a normal subgroup of A_n.
- But we know that A_n is a simple group.
- It follows that $G_{k-1} = \{e\} = G_0$.
- However, the factor group G_k/G_{k-1}, which is A_n itself, is not abelian, a contradiction arises.

$\mathbf{n = 4}$ A solvable series is given by

$$\{\text{Id}\} < V < A_4 < S_4$$

(V is the Klein-4-group).

$\mathbf{n = 3}$ A solvable series is given by

$$\{\text{Id}\} < \langle(1\ 2\ 3)\rangle < S_3.$$

$\mathbf{n = 1, 2}$ Exercise.

(c)
- Let $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\} \subseteq G$.
- It is the kernel of the surjective homomorphism

$$\phi : G \to \mathbb{R}^\times \times \mathbb{R}^\times : \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d).$$

- Hence $H \triangleleft G$.
- Now we show that the series

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} < H < G$$

is a solvable series.
  - $H \Big/ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq H \simeq \mathbb{R}$ which is abelian.
  - $G/H \simeq \mathbb{R}^\times \times \mathbb{R}^\times$ which is also abelian.
- It follows that G is solvable.

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2. Find a composition series for each of the following groups.

   (a) D_n

   (b) S_n

   (c) a finite abelian group

Solution. In the following, we use frequently the fact that $\mathbb{Z}_p$ is simple whenever p is prime.

(a)
- Consider first the series $\{\text{Id}\} < \langle \sigma \rangle < D_n$.
- We have $D_n/\langle \sigma \rangle \simeq \mathbb{Z}_2$ being simple.
- It remains to find a composition series for $\langle \sigma \rangle$.
- Let $n = p_1 p_2 p_3 \cdots p_k$ be a prime factorisation, here the primes p_i are not necessarily distinct.
- Then the series

$$\{\text{Id}\} = \langle \sigma^{p_1 \cdots p_k} \rangle < \langle \sigma^{p_2 \cdots p_k} \rangle < \langle \sigma^{p_3 \cdots p_k} \rangle < \cdots < \langle \sigma^{p_k} \rangle < \langle \sigma^1 \rangle = \langle \sigma \rangle$$

is a composition series of $\langle \sigma \rangle$.

(b)

$\mathbf{n \geq 5}$ $\{\text{Id}\} < A_n < S_n$.

$\mathbf{n = 4}$ $\{\text{Id}\} < \langle (1\ 2)(3\ 4) \rangle < V < A_4 < S_4$. ($V$ is the Klein-4-group).

$\mathbf{n = 3}$ $\{\text{Id}\} < \langle (1\ 2\ 3) \rangle < S_3$.

$\mathbf{n = 1, 2}$ Exercise.

(c)
- By FTFGAG, every finite abelian group is isomorphic to a finite product of finite cyclic groups.
- But in (a), we have seen how to construct a composition series for a finite cyclic group.
- This construction can actually be extended to the case of a finite product of finite cyclic groups.
- This task is left to you as an exercise.

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3. Let M and N be normal subgroups of a group G that are both soluble. Show that MN is soluble. Deduce that a finite group G has a largest normal subgroup S which is soluble.

Solution. Since $M, N \triangleleft G$, it follows that $MN \triangleleft G$. Note that $M \triangleleft MN$ and the Second Isomorphism Theorem gives $MN/M \cong N/(M \cap N)$. Now $N/(M \cap N)$ is a homomorphic image of the soluble group N, so this quotient is soluble. Hence MN has a soluble normal subgroup M with soluble quotient MN/M. We deduce that MN is soluble.

Let $S \triangleleft G$ be a soluble normal subgroup of G of largest possible order. (Certainly $\{e\}$ satisfies the property of being soluble and normal, so it is possible to take one whose order is as large as possible.) Now let $N \triangleleft G$ and suppose N is soluble. By the above, $SN \triangleleft G$ and SN is soluble. Since S is the largest such subgroup, $|SN| \leq |S| \leq |SN|$. But $S < SN$, so $S = SN$. Hence $N < SN = S$ and S contains all soluble normal subgroups of G. ◀

4. Let M and N be normal subgroups of a group G such that G/M and G/N are both soluble. Show that $G/(M \cap N)$ is soluble. Deduce that a finite group G has a smallest normal subgroup R such that the quotient G/R is soluble.

Solution. Define $\phi : G \to G/M \times G/N$ by $\phi(g) = (Mg, Ng)$. Then ϕ is a homomorphism and $ker\phi = g \in G|(Mg, Ng) = (Me, Ne) = M \cap N$. Hence by the First Isomorphism Theorem, $G/(M \cap N) \cong im\phi < G/M \times G/N$. Since G/M and G/N are soluble, $G/M \times G/N$ is soluble, and then $im\phi$ is a subgroup of a soluble group, so is soluble. Hence $G/(M \cap N)$ is soluble.

G is a normal subgroup of itself such that the quotient $G/G = \{e\}$ is soluble. Choose a normal subgroup R of G of smallest possible order such that G/R is soluble. Let M be any normal subgroup of G such that G/M is soluble. Then the above tells us that $G/(M \cap R)$ is soluble. Therefore $|R| < |M \cap R|$ by choice of R, and $M \cap R < R$, so $R = M \cap R < M$. Hence R is contained in all other normal subgroups of M of G whose quotient is soluble. ◀

5. It is given that if S is a simple finite group and $|S|$ is not divisible by 3 or 5, then $|S|$ is a prime. Let G be a finite group with order larger than 1 and relatively prime to 15. Show that G has a nonidentity abelian normal group.

Solution. Let M minimal normal in G. If G is not simple, then we are done. Otherwise G is simple, this G is of prime order by the fact, so we can take $M = G$. Let N be the maximal normal in M. So M/N is finite and simple (Consider the canonical homomorphism f from M to M/N and note that the preimage of normal subgroup of M/N must contain N). By the fact that M/N is of prime order and hence abelian. Hence $[M, M] < N$, in particular $[M, M] < M$. So $[M, M]$ is normal in G as $gxyx^{-1}y^{-1}g^{-1} = gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \in [M, M]$. It forces that $[M, M]$ is trivial from the minimality. ◀