THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 3
Date: 26th September 2019

1. Show that $\mathbb{Q}$ is not finitely generated.

Solution.

- We prove by contradiction. Suppose $\mathbb{Q}$ is finitely generated.
- Let $x_1, x_2, \ldots, x_n$ be a set of generators of $\mathbb{Q}$.
- For each $i = 1, 2, \ldots, n$, write $x_i = \frac{p_i}{q_i}$ where p_i, q_i are integers and q_i is nonzero.
- Let Q be a nonzero integer divisible by each of the q_i's (e.g. let $Q = q_1 q_2 \cdots q_n$).
- Consider $\frac{1}{2Q} \in \mathbb{Q}$.
- More explicitly, there are integers $k_1, k_2, \ldots, k_n$ such that

$$\frac{1}{2Q} = k_1 \frac{p_1}{q_1} + k_2 \frac{p_2}{q_2} + \cdots + k_n \frac{p_n}{q_n}.$$

- Multiplying both sides by Q, we get

$$\frac{1}{2} = k_1 p_1 \left(\frac{Q}{q_1}\right) + k_2 p_2 \left(\frac{Q}{q_2}\right) + \cdots + k_n p_n \left(\frac{Q}{q_n}\right).$$

- Note that the RHS is an integer while the LHS is not, hence a contradiction.
- We conclude that $\mathbb{Q}$ is not finitely generated.

◀

2. Find the maximum possible order for some element of $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$

Solution. $120 = l.c.m(8, 10, 24)$ ◀

3. Are the groups $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ isomorphic?

Solution. No, they do not have the same kind of decomposition as in the fundamental Theorem of finitely generated abelian groups. $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. But on the other hand $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

◀

4. Let U_{24} be a group of integers less than 24 and relatively prime to 24 with multiplication modulo 24. Find a product of cyclic groups which is isomorphic to U_{24}. Can you find a corresponding group isomorphism?

Solution. Note that $U_{24} = \{\pm 1, \pm 5, \pm 7, \pm 11\}$ and it is a finitely generated abelian group of order 8. By the fundamental Theorem of finitely generated abelian groups, it is isomorphic to one of the following three groups:

$$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

U_{24} is isomorphic to $4\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as each element in U_{24} is of order 2.

One possible answer is: Define $f : U_{24} \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by

$$f(1) = (0,0,0), f(-1) = (1,1,1), f(5) = (1,0,0), f(-5) = (0,1,1),$$

$$f(7) = (0,1,0), f(-7) = (1,0,1), f(11) = (1,1,0), f(-11) = (0,0,1).$$

◀

5. For a finite group G, let $d_2(G)$ be the number of elements having order 2 in G. Let $(G,+)$ be a finite abelain group, and let $S = \sum_{x \in G} x$. Show that if $d_2(G) \neq 1$, then $S = 0_G$.

Solution. Let $G_2 = \{x \in G | 2x = 0_G\}$. Note that it is abelain finite subgroup of G (Closed: $\forall x, y \in G_2, 2(x+y) = x+y+x+y = x+x+y+y = 0_G + 0_G = 0_G$, identity: $0_G \in G_2$ as $2(0_G) = 0_G$, inverse: $\forall x \in G_2, 2x = 0_G \Rightarrow -2x = 0_G \Rightarrow -x \in G_2$.). Every nonzero element of G_2 has order 2, so by the fundamental Theorem of finitely generated abelian groups,

$$G_2 \cong \mathbb{Z}_2^k$$

for some non-negative integer k.

Consider the involution $\iota : G \to G$ given by $\imath(x) = -x$. The fixed points of ι - i.e., the elements $x \in G$ such that $\iota(x) = x$ – are precisely the elements of G_2. Thus the elements of $G \setminus G_2$ occur in pairs of distinct elements $x, -x$, so $\sum_{x \in G \setminus G_2} x = 0$. In other words,

$$S = \sum_{x \in G} x = \sum_{x \in G_2} x.$$

We are reduced to the case $G_2 \cong \mathbb{Z}_2^k$.

When $k = 0$, $G_2 = \{0_G\}$. In this case, $d_2(G) = 0$ and

$$S = \sum_{x \in G} x = \sum_{x \in G_2} x = \sum_{x \in \{0_G\}} x = 0_G.$$

When $k = 1$, $G_2 \cong \mathbb{Z}_2$. In this case, $d_2(G) = 1$.

When $k \geq 2$, $d_2(G) \geq 3$, so we wish to show $S = 0_G$.

For each $1 \leq i \leq k$, half of the elements of $\mathbb{Z}_2^k$ have ith coordinate $0 \in \mathbb{Z}_2$; the other half have ith coordinate $1 \in \mathbb{Z}_2$. So the sum of the ith coordinates of the elements of $\mathbb{Z}_2^k$ is $2^k/2 = 2^{k-1} = 0 \in \mathbb{Z}_2$, since $k \geq 2$. Every coordinate of S equals 0_G, so $S = 0_G$.

◀