THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2019-20
Tutorial 2
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1. Determine which of the following subgroups are normal.

   (a) The Klein-4-group $< D_4$

   (b) $D_4 < S_4$

   (c) $S_n < S_{n+1}$

   (d) $SL(n, \mathbb{R}) < GL(n, \mathbb{R})$

   (e) $GL(n, \mathbb{R}) < GL(n+1, \mathbb{R})$

Solution. (a) Yes. It follows from the fact that every subgroup of index 2 of a group is normal.
(How about in S_4? Yes, we can check it directly. How about in A_4? Yes, since it is a normal subgroup of S_4.)

(b) No. Take $g = (1\ 2\ 3) \in S_4$ and $h = (1\ 2\ 3\ 4) \in D_4$. Then $ghg^{-1} = (1\ 4\ 2\ 3) \notin D_4$.

(c) No, unless $n = 1$. For $n \geq 2$, take $g = (1\ n+1) \in S_{n+1}$ and $h = (1\ 2) \in S_n$. Note that $n + 1 \neq 2$. Then $ghg^{-1} = (2\ n+1) \notin S_n$.

(d) Yes. Let $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^\times$ be the map given by $\phi(X) = \det(X)$, for all $X \in GL(n, \mathbb{R})$. The map ϕ is a group homomorphism. Note that its kernel is $SL(n, \mathbb{R})$, so $SL(n, \mathbb{R})$ is a normal subgroup of $GL(n, \mathbb{R})$ as the kernel is always a normal subgroup.

(e) No. Consider the map $\phi_n : S_n \to GL(n, \mathbb{R})$ (where $\mathbb{R}$ can actually be replaced by any field) defined by letting $\phi_n(\sigma)$ be the matrix permutating the coordinate vectors e_i to $e_{\sigma(i)}$ for $i = 1, 2, \ldots, n$. It can be shown that ϕ_n is a homomorphism and $\phi_{n+1}^{-1}(GL(n, \mathbb{R})) = S_n$, so if $GL(n, \mathbb{R})$ is normal in $GL(n+1, \mathbb{R})$, then S_n is also normal in S_{n+1} but it is not true as shown in (c). In fact the preimage of a normal subgroup under a group homomorphism is also normal.

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2. True or false: $K \lhd H \lhd G \Rightarrow K \lhd G$.

Solution. No. $\langle s \rangle \lhd \langle r^2, s \rangle \cong$ Klein-4-group $\lhd D_4$ but $\langle s \rangle$ is not a normal subgroup of D_4.

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3. Recall: Let H be a subgroup of G. Then

$$aHa^{-1} \subseteq H \text{ for all } a \in G \Longrightarrow aHa^{-1} = H \text{ for all } a \in G.$$

What if we only consider a single element a of G:
Is it true that $(aHa^{-1} \subseteq H \Longrightarrow aHa^{-1} = H)$ (a) if G is finite? (b) if G is infinite?

Solution. (a) Yes. Suppose $aHa^{-1} \subseteq H$. Then the map $\phi : G \to G : x \mapsto axa^{-1}$, when restricted to H, gives rise to the map $\phi' : H \to H$. It is injective because ϕ is. Since G is finite, ϕ is also surjective and hence bijective. It follows that for every $h \in H$, there is $x \in H$ such that $\phi'(x) = h$, that is $axa^{-1} = h$. This shows that $aHa^{-1} = H$.

(b) No, because of the following counter-example:
Let $G = (\mathbb{Z} \times \mathbb{Q}, *)$ where $*$ is not the direct product but defined by

$$(m, s) * (n, t) = (m + n, 2^n s + t)$$

for any $m, n \in \mathbb{Z}$ and $s, t \in \mathbb{Q}$.
I leave to you the task of showing that G is indeed a group. Note that the identity of G is $(0, 0)$ and the inverse of $(m, s) \in G$ is $(-m, -2^{-m}s)$. Now let $H = \{0\} \times \mathbb{Z}$ and $a = (-1, 0)$. Then H is a subgroup of G and $aHa^{-1} = \{0\} \times 2\mathbb{Z}$ which is a proper subgroup of H. This shows that $aHa^{-1} \neq H$.

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4. Show that S_n $(n \geq 3)$ has no normal subgroup of order 2.

Solution.

Let $H < S_n$ have order 2. Then $H = \{\text{Id}, \sigma\}$ where $\sigma \neq \text{Id}$ is a permutation. If H is normal in S_n, then for any $\tau \in S_n$, $\tau\sigma\tau^{-1} \in H$. The last expression cannot be the identity, for otherwise $\sigma = \text{Id}$. Hence $\tau\sigma\tau^{-1} = \sigma$, or equivalently $\tau\sigma = \sigma\tau$. It implies that σ commutes with every element of S_n. It is impossible for $n \geq 3$. (why?) It follows that no subgroup of order 2 of S_n is normal.

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5. Let p be the smallest prime dividing the order of group G. If H is a subgroup of G with index p then H is normal.

Solution. Let H be a subgroup of index p where p is the smallest prime that divides $|G|$. Let $\phi : G \times G/H \to G/H$ the map given by $\phi(x, gH) = xgH$. This induces a homomorphism $\phi^* : G \to S_p$. Note that the kernel is contained in H. Denote the kernel by K. Then $G\backslash K$ is a subgroup of S_p, so $|G/K||p!$. Since

$$|G/K| = |G/H||H/K| = p|H/K|,$$

we have p divides $|G/K|$.

We claim that $|G/K| = p$. $|G/K|$ cannot have prime factor less than p for p is the smallest prime dividing $|G|$. There is no prime larger than p dividing $|G/K|$ as $|G/K|$ divides $p!$.

So it forces that $|H/K| = 1$ saying that H is indeed a kernel which is always normal.

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6. Give a list of all non-isomorphic abelian groups of order 180 but not isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_{30}$.

Solution. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5, \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5, \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$ ◀