# MATH 2060 Mathematical Analysis II
## Tutorial Class 6

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**Theorem 1** (The second fundamental theorem of Calculus)**.** *Suppose that the function* $f : [a,b] \to \mathbb{R}$ *is continuous. Then* $F(x) = \int_a^x f$ *satisfy*

$$F'(x) = f(x) \ , \forall x \in (a,b).$$

Problems :

1. (a) Prove the Second Fundamental Theorem of Calculus.

   (b) State and prove the Integration by Parts formula.

   (c) Let $f : [0,+\infty) \to \mathbb{R}$ be a continuous function. Define

   $$F(x) = \int_0^x f(x^2 + y)dy.$$

   Find $F'(x)$.

2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function at which $f(x) \geq 0$. Show that

   $$\lim_{n\to\infty} (\int_a^b f^n)^{\frac{1}{n}} = \sup\{f(x) : x \in [a,b]\}.$$

3. Suppose that the function $f : [a,b] \to \mathbb{R}$ is continuous and it is twice differentiable on $(a,b)$. Prove that there is a point $\eta \in (a,b)$ at which

   $$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a) + f(b)\right] - \frac{(b-a)^3}{12} f''(\eta).$$

4. (a) Suppose $f : [0,+\infty) \to \mathbb{R}$ is continuous and strictly increasing, and that $f : (0,+\infty] \to \mathbb{R}$ is differentiable and $f(0) = 0$. Prove that for all $a > 0$,

   $$\int_0^a f + \int_0^{f(a)} f^{-1} = af(a).$$

   (b) If $f$ satisfies the assumption above, prove that for all $a > 0$ and $b > 0$,

   $$\int_0^a f + \int_0^b f^{-1} \geq ab$$

   (c) If $a$ and $b$ are two non-negative real number, $p$ and $q$ are positive real number such that $\frac{1}{p} + \frac{1}{q} = 1$, show that

   $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$