MATH 2060 Mathematical Analysis II
Tutorial Class 2

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1. Evaluate the Limits:
   (a) $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$
   (b) $\lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x}$

2. Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

   Give an example where this limit exists, but the function is not twice differentiable at a.

3. Suppose the function $f : (-1,1) \to R$ has n derivatives, and $f^{(n)} : (-1,1) \to \mathbb{R}$ is bounded. Prove that there exists $M > 0$ such that $|f(x)| \le M|x|^n, \forall x \in (-1,1)$ if and only if $f(0) = f'(0) = ... = f^{(n-1)}(0) = 0$.

4. (a) State the Taylor's theorem.
   (b) Prove that $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$ for all $x \in (0, \pi]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function satisfying
   (i) $f(x) > f(0)$ for all $x \ne 0$, and
   (ii) there exists $M > 0$ such that $|f^{(n)}(x)| \le M$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.
   (a) Show that there exists $n \in \mathbb{N}$ such that $f^{(n)}(0) \ne 0$.
   (b) Prove that there exists an even number $2k$ such that $f^{(2k)}(0) > 0$.
   (c) Prove that there exists $\delta > 0$ such that $f'(y) < 0 < f'(x)$ for all x, y with $-\delta < y < 0 < x < \delta$.