MATH 2060 Mathematical Analysis II
Tutorial Class 2
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1. (a) State Mean Value Theorem.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function differentiable on $\mathbb{R}$. prove that if f' is bounded on $\mathbb{R}$, then f is uniformly continuous.
   (c) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function such that f' is bounded on (a, b). Show that f is bounded function.
   (d) If f is uniform continuous on $[a, b]$ and differentiable on (a, b) , is f' bounded on (a, b) ? Prove or disprove it.

2. (a) Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). If $f' > 0$ on (a, b), show that f is strictly increasing on $[a, b]$.
   (b) Prove that $\tan x > x > \sin x > \frac{2}{\pi}x$ for all $x \in (0, \frac{\pi}{2})$.
   (c) Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on (a, b). Show that if $\lim_{x\to a} f'(x) = A$, then $f'(a)$ exists and equals to A.

3. Let $f : \mathbb{R} \to \mathbb{R}$be a twice differentiable function. Suppose

$$f(x) \le 0 \quad \text{and} \quad f''(x) \ge 0 \quad , \forall x \in \mathbb{R}.$$

   Prove that f is constant function.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a differentaible function on $(0, +\infty)$ and assume $\lim\limits_{x\to\infty} f'(x) = b$.
   (a) Show that for any $h > 0$, we have $\lim\limits_{x\to\infty} \frac{f(x + h) - f(x)}{h} = b$.
   (b) Show that if $f(x) \to a$ as $x \to \infty$, then $b = 0$.
   (c) Show that $\lim\limits_{x\to\infty} \frac{f(x)}{x} = b$.

5. (a) State the Taylor's theorem.
   (b) Prove that $\sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$ for all $x \in (0, \pi]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function satisfying
   (i) $f(x) > f(0)$ for all $x \ne 0$, and
   (ii) there exists $M > 0$ such that $|f^{(n)}(x)| \le M$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.
   (a) Show that there exists $n \in \mathbb{N}$ such that $f^{(n)}(0) \ne 0$.
   (b) Prove that there exists an even number $2k$ such that $f^{(2k)}(0) > 0$.
   (c) Prove that there exists $\delta > 0$ such that $f'(y) < 0 < f'(x)$ for all x, y with $-\delta < y < 0 < x < \delta$.