MATH 2060 Mathematical Analysis II
Tutorial Class 12

1. (a) Show that $f(x) = \sum_{n=1}^{\infty} \frac{\cos 3^n x}{2^n}$ is a continuous function on $\mathbb{R}$.

   (b) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{e^{nx}}{n!}$ is a continuous function on $\mathbb{R}$ but the convergence is non-uniform.

   (c) Show that $f(x) = \sum_{n=1}^{\infty} \frac{n^{10}}{x^n}$ is a differentiable function on $(1, \infty)$.

2. Let $\{a_n\}$ be a sequence such that $\sum_{n=1}^{\infty} n|a_n|$ converge. Show that $f(x) = \sum_{n=1}^{\infty} a_n \sin nx$ converge on $\mathbb{R}$ and $f'(x) = \sum_{n=1}^{\infty} na_n \cos nx$.

3. Show that the convergence of $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is not uniform on $[0, 1]$.

4. (a) State the Cauchy-Hadmand Theorem for power series.

   (b) Suppose a power series $\sum a_n x^n$ converge at some $x_0 \in \mathbb{R}$. Show that it converge absolutely for all $|x| < |x_0|$.

   (c) Suppose a power series converge absolutely at some $c \in \mathbb{R}$, show that it converge uniformly on the interval $[-c, c]$.

5. Find the radius of convergence R of the following series:
   (i) $\sum \frac{2^n}{n^2} x^n$ (ii) $\sum n! x^n$ (iii) $\sum \frac{n!}{(2n)!} x^n$ (iv) $\sum \frac{(-1)^n + 2^n}{3^n} x^n$.

6. (a) Prove that for all $x \in (-1, 1)$,

      i. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$,

      ii. $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ and

      iii. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

   (b) Find the value of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

7. Let $f_n : [a, b] \to \mathbb{R}$ such that $\sum f_n$ converge uniformly on (a, b). Suppose $\lim_{x\to a^+} f_n(x) = c_n \in \mathbb{R}$. Show that $\sum c_n$ converge and

$$\lim_{x \to a^+} \sum f_n(x) = \sum c_n.$$

past paper question:

Suppose the series $\sum a_n x^n$ has radius of convergence one. Let $f(x) = \sum a_n x^n$, $x \in (-1, 1)$. If $[a, b] \subset (0, 1)$ and $f_n(x) \doteq f(x - \frac{1}{n}), x \in [a, b]$, show that $f_n \to f$ uniformly on $[a, b]$.