MATH 2060 Mathematical Analysis II
Tutorial Class 11

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1. Show that the convergence of $\sum_n \sqrt{a_n a_{n+1}}$ does not imply the convergence of $\sum_n a_n$, even if $a_n > 0, \ \forall n \in \mathbb{N}$.

2. If $\{a_n\}$ is a decreasing sequence of strictly positive numbers and if $\sum_n a_n$ is convergent, show that $\lim_{n\to} na_n = 0$.

3. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and
$$\limsup_n |\frac{a_{n+1}}{a_n}| = L.$$
   (a) Prove that if $L < 1$, then the series $\sum a_n$ converges absolutely.
   (b) If $\liminf_n |\frac{a_{n+1}}{a_n}| > 1$, show that the series diverges.

4. If
$$\limsup_n |a_n|^{1/n} = L.$$
   (a) Prove that if $L < 1$, then the series $\sum a_n$ converges absolutely.
   (b) Prove that if $L > 1$, then the series $\sum a_n$ diverge.
   (c) If $a_n > 0$, show that
$$\limsup_n a_n^{1/n} \ \leq \ \limsup_n |\frac{a_{n+1}}{a_n}|.$$

5. Determine the convergence of following series.
   (a) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ (b) $\sum_{n=1}^{\infty} (1 - \cos \frac{1}{n})$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{2n+3}$
   (d) $\sum_{n=1}^{\infty} \frac{1+\log^2 n}{n \log^2 n}$ (e) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ (f) $\sum_{n=1}^{\infty} \frac{\log n}{n+\log n}$

6. Let A be the set of positive integers which do not contain the digit 9 in the decimal expansion. Prove that
$$\sum_{a\in A} \frac{1}{a} \ \text{ exists.}$$

7. Find the value of $a \in \mathbb{R}$ such that the series
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\frac{1}{n}\right)^a$$
exists.