Suggested solution of HW7

P286, 1 a: $\sum f_n = \sum \dfrac{1}{x^2+n^2}$, $x \in \mathbb{R}$. For any $x \in \mathbb{R}$, we have

$$0 \le \frac{1}{x^2+n^2} \le \frac{1}{n^2} \ , \forall\, n \in \mathbb{N}.$$

Thus, by Weierstrass M-test, $\sum f_n$ is uniformly convergent on $\mathbb{R}$.

P286, 1 c: $\sum f_n = \sum \sin\left(\dfrac{x}{n^2}\right)$, $x \in \mathbb{R}$. Since we have for all $x \in \mathbb{R}$,

$$0 \le |\sin(\frac{x}{n^2})| \ \le \ |\frac{x}{n^2}| \ ,\forall\, n \in \mathbb{N}.$$

Thus for all $x \in \mathbb{R}$, $\sum f_n(x)$ converge. But since $f_n(x) = \sin\,(\frac{x}{n^2})$ does not converge uniformly to 0 on $\mathbb{R}$, the convergence is non-uniform. It can be checked by choosing $x_k = k^2$, then

$$\sin(\frac{x_k}{k^2}) = \sin 1 > 0 \quad \forall\, k \in \mathbb{N}.$$

P286, 1 d: $\sum f_n = \sum \dfrac{1}{x^n+1}$, $x \neq 0$. For each $|x| < 1$,

$$\frac{1}{x^n+1} \to 1 \quad \text{as } n \to \infty.$$

So the series is not convergent on $(-1, 1)$. The series is not well-defined at $x = -1$. Since $f_n(1) = 1/2$, the series is not convergent at $x = 1$.
For each $|x| > 1$, since

$$\left|\frac{x^n+1}{x^{n+1}+1}\right| \to \frac{1}{|x|} < 1 \quad \text{as } n \to \infty.$$

The series converge. The convergence is non-uniform since we can choose a sequence $\{x_n = 1 + 1/n\}$ at which $f_n(x_n) \to \frac{1}{e+1} \neq 0$.

P286, 6 a: By cauchy-Hadamard Theorem, radius of convergence $R = \dfrac{1}{\limsup |a_n|^{1/n}}$. So for $a_n = \frac{1}{n^n}$, $R = +\infty$.

P286, 6 c: For $a_n = \frac{n^n}{n!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \to e \text{ as } n \to \infty.$$

Since $\lim_n |a_n|^{1/n} = \lim_n \left|\dfrac{a_{n+1}}{a_n}\right|$, if right hand side limit exists. Thus, $R = \dfrac{1}{e}$.

P286, 6 e: For $a_n = \dfrac{(n!)^2}{(2n)!}$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!(n+1)!}{n!\; n!} \cdot \frac{(2n)!}{(2n+2)!} \to \frac{1}{4}.$$

As argue before, $R = 4$.

P287, 16 : It is known that

$$\int_0^x \frac{1}{1+y}\ dy = \ln(1+x)\ ,\ \forall x \in (-1,1).$$

Also, by the result of geometric series, we have

$$\frac{1}{1+y} = \sum_{n=0}^{\infty}(-1)^n y^n\ , \forall\ y \in (-1,1).$$

If $1 > x > 0$, take $I = [0,x] \subset (-1,1)$ at which it contains x. Since

$$|(-1)^n y^n| \le a^n < 1\ \forall n \in \mathbb{N}.$$

By M-test, $\sum(-1)^n y^n$ converge uniformly on I. Thus,

$$\begin{aligned}\int_0^x \frac{1}{y+1}\ dy = \int_0^x \sum_{n=0}^{\infty}(-1)^n y^n\ dy &= \sum_{n=0}^{\infty}(-1)^n \int_0^x y^n\ dy \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}.\end{aligned}$$

The case of $x < 0$ is similar.

P287, 17 : If $x \in (-1,1)$, we consider the case of $x > 0$ first. Noted that

$$\arctan x = \int_0^x \frac{1}{t^2+1}\ dt.$$

By mean of geometric series, we have for all $x \in (-1,1)$,

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{x^2+1}.$$

Given $x \in [0,-1)$ fixed, for all $t \in [0,x]$,

$$|t^{2n}| \le |x|^{2n}$$

at which $\sum x^{2n}$ converge. By M-test, $\sum(-1)^n t^{2n}$ converge uniformly on $[0,x]$ which implies

$$\begin{aligned}\arctan x = \int_0^x \frac{1}{t^2+1}\ dt &= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\ dt \\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}\ dt \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

P287, 17 : Formally, we have for all $x \in \mathbb{R}$,

$$\begin{aligned}\int_0^x \exp(-t^2)dt &= \int_0^x \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!}\,dt \\ &= \sum_{n=0}^{\infty} \int_0^x \frac{(-t^2)^n}{n!}\,dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}.\end{aligned}$$

The first equality holds since the radius of convergence of exp function is $+\infty$. It remains to check wether the second equality holds or not. It suffices to check that $\sum \frac{(-1)^n t^{2n}}{n!}$ converge uniformly on $[0, x]$ (or $[x, 0]$) for each $x > 0$ (or $x < 0$). Since for any $t \in [0, x]$, $n \in \mathbb{N}$,

$$\left|\frac{t^{2n}}{n!}\right| \leq \left|\frac{x^{2n}}{n!}\right|.$$

And $\sum \frac{x^{2n}}{n!}$ converges. It can be checked by ratio test as

$$|\frac{n!}{(n+1)!} \cdot \frac{x^{2n+2}}{x^{2n}}| \to 0 \text{ as } n \to \infty.$$