THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Special Tutorial 1 (February 13)

The following were discussed in the tutorial this week.

1. (HW 3 Q4)

   Let $f : [a,b] \to \mathbb{R}$. The Lower Envelope of f is defined as the function $\underline{f} : [a,b] \to [-\infty, \infty]$ given by

   $$\underline{f}(x) := \sup\{g_\delta(x) : \delta > 0\} \quad \text{for all } x \in [a,b],$$

   where $g_\delta(x) := \inf\{f(y) : y \in [a,b], |x-y| < \delta\}$.

   (a) Show that $\underline{f} \leq f$ pointwisely on $[a,b]$. Moreover, show that for all $x \in [a,b]$, $\underline{f}(x) = f(x)$ if and only if f is l.s.c at x.

   (b) Show that if f is bounded, then $\underline{f}$ is l.s.c.

   (c) Show that if ϕ l.s.c on $[a,b]$ such that $\phi \leq f$ on $[a,b]$, then $\phi \leq \underline{f}$.

2. (HW3 Q5) Let $f : [a,b] \to [m,M]$. For each $P \in \text{Par}[a,b]$, let $u(f;P)$ and $U(f;P)$ denote the lower/upper Riemann-sum functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}\ \forall\, n$ and $\|P_n\| \to 0$ ($\|P\|$ is the max subinterval length of P). Show that, $\forall\, x \in [a,b] \setminus A$

   $$\lim_n \left(u(f;P_n)\right)(x) = \underline{f}(x) \quad \text{and} \quad \lim_n \left(U(f;P_n)\right)(x) = \overline{f}(x),$$

   where A denotes the union of all end-points of $P_n\ \forall\, n$.

3. (a) Recall the definition of Borel σ-algebra $\mathcal{B}$, the smallest σ-algebra that contains all open sets in $\mathbb{R}$.

   (b) Note that $\mathcal{B}$ can also be generated by $\{(a,b) : a < b\}$ or $\{[a,b] : a < b\}$.

4. (a) Recall the definition of G_δ-sets and F_σ-sets.

   (b) Briefly explain that any closed set is G_δ, and any open set is F_σ.

   (c) Give an example of an F_σ-set that is not G_δ.

   (d) Give an example of a Borel set that is neither F_σ nor G_δ.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and B be a Borel subset of $\mathbb{R}$.

   (a) Is $f^{-1}(B)$ Borel?

   (b) Is $f(B)$ Borel? What if we further assume that f is injective?

6. Let $f : \mathbb{R} \to \mathbb{R}$. Show that the set of continuity point of f is a G_δ-set.