THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 4 (March 15)

The following were discussed in the tutorial this week.

Definition 1 (Vitali Cover). *Let $E \subseteq \mathbb{R}$. A collection $\mathcal{U}$ of nondegenerate intervals is said to be a **Vitali cover** of E if*

$$\forall\, x \in E,\ \forall\, \varepsilon > 0,\ \exists\, I \in \mathcal{U} \text{ such that } x \in I \text{ and } \ell(I) < \varepsilon,$$

$$\text{i.e. } \forall\, x \in E,\ \inf\{\ell(I) : x \in I \in \mathcal{U}\} = 0.$$

Remark. *If $\mathcal{U}$ is a Vitali cover of E and G is an open set, then $\mathcal{V} := \{I \in \mathcal{U} : I \subseteq G\}$ is a Vitali cover of $E \cap G$.*

Lemma 1 (Vitali Covering Lemma). *Let $m^*(E) < \infty$. Let $\mathcal{U}$ be a Vitali cover of E. Then $\forall\, \gamma > 0$, $\exists$ disjoint $\{I_1, \ldots, I_N\} \subseteq \mathcal{U}$ such that*

$$m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \gamma.$$

Remark.

1. *E need not be measurable.*
2. *This version of Vitali lemma does not hold when $m^*(E) = \infty$.*
3. *In the proof, we actually find a countable disjoint subcollection $\{I_n\}_{n=1}^{\infty} \subseteq \mathcal{U}$ such that for some $N \in \mathbb{N}$,*

$$E \subseteq \bigcup_{n=1}^{N} I_n \cup \bigcup_{n=N+1}^{\infty} \widehat{I}_n,$$

where $\widehat{I}_n$ is the interval whose centre is the same as that of I_n and $\ell(\widehat{I}_n) = 5\ell(I_n)$, and

$$\sum_{n=N+1}^{\infty} m^*(I_n) < \gamma,$$

Using the Vitali lemma, we readily deduce the following.

Theorem 2 (Vitali Covering Theorem). *Let $m^*(E) < \infty$. Let $\mathcal{U}$ be a Vitali cover of E. Then there exists a countable disjoint subcollection $\{I_n\}_{n=1}^{\infty} \subseteq \mathcal{U}$ such that*

$$m^*\left(E \setminus \bigcup_{n=1}^{\infty} I_n\right) = 0.$$

Remark. *The assumption "$m^*(E) < +\infty$" can be dropped.*

Exercise 1. *Let E be a (not necessarily countable) union of nondegenerate intervals (open, closed, half open and half closed, infinite, etc). Show that E is measurable.*

Exercise 2. *Let $E \subseteq \mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function (not necessarily measurable).*

For $\alpha > 0$, define

$$E_\alpha = \{x \in E : f'(x) \text{ exists and } |f'(x)| < \alpha\}.$$

Show that $m^*(f(E_\alpha)) \leq \alpha m^*(E_\alpha)$.