

Examples of bounded linear operator and its norm

Eg 1: Let $K(x, y)$ be a continuous function on $[0, 1] \times [0, 1]$

Define $T: C[0, 1] \rightarrow C[0, 1]$ by

$$Tu(x) = \int_0^1 K(x, y)u(y)dy$$

Show that T is a bounded linear operator

$$\text{and } \|T\| = \max_{x \in [0, 1]} \int_0^1 |K(x, y)| dy$$

Pf: Set $k(x) = \int_0^1 |K(x, y)| dy$. Since K is continuous, so is k

Then $Tu \in C[0, 1]$. It is obvious that T is linear

$$\begin{aligned} \text{Since } \|Tu\|_{C[0, 1]} &= \max_{x \in [0, 1]} |Tu(x)| = \max_{x \in [0, 1]} \left| \int_0^1 K(x, y)u(y)dy \right| \\ &\leq \max_{y \in [0, 1]} |u(y)| \max_{x \in [0, 1]} \int_0^1 |K(x, y)| dy = \|k\|_{C[0, 1]} \|u\|_{C[0, 1]} \end{aligned}$$

Therefore T is bounded and $\|T\| \leq \|k\|_{C[0, 1]}$

Now, it suffices to show that $\|T\| \geq \|k\|_{C[0, 1]}$

Since k is continuous in $[0, 1]$, then $\exists x_0 \in [0, 1]$ s.t. $k(x_0) = \|k\|_{C[0, 1]}$

$$\text{Set } \varphi(y) = \text{sgn } K(x_0, y) = \begin{cases} 1 & \text{if } K(x_0, y) > 0 \\ -1 & \text{if } K(x_0, y) < 0. \end{cases}$$

By Lusin's Thm, $\exists \{u_n\} \subset C[0, 1]$ (w.l.o.g. with $|u_n(y)| \leq 1$) s.t. $u_n \rightarrow \varphi$

$$\begin{aligned} \text{Thus, } (\|k\|_{C[0, 1]} =) k(x_0) &= \int_0^1 K(x_0, y)\varphi(y)dy \\ &= \lim_{n \rightarrow \infty} \int_0^1 K(x_0, y)u_n(y)dy \quad \text{by L.D.C.T.} \\ &= \lim_{n \rightarrow \infty} Tu_n(x_0) \leq \overline{\lim_{n \rightarrow \infty}} \|Tu_n\|_{C[0, 1]} \\ &\leq \overline{\lim_{n \rightarrow \infty}} \|Tu_n\|_{C[0, 1]} \\ &\leq \overline{\lim_{n \rightarrow \infty}} \|T\| \|u_n\|_{C[0, 1]} \\ &\leq \|T\| \end{aligned}$$

Ex 2. Let $A = (a_{ij})$ ($i, j = 1, 2, \dots$) be an infinite matrix.

Define $A: x \rightarrow Ax$ by $y = Ax$ such that $y_i = \sum_j a_{ij} x_j$, $\forall x = \{x_j\}, y = \{y_i\}$

Then: (i) If $C = \sup_i \sum_j |a_{ij}| < +\infty$, then A is a bounded linear operator on ℓ^1 and $\|A\|_1 = C$

(ii) If $C = \sup_i \sum_j |a_{ij}| < +\infty$, then A is a bounded linear operator on ℓ^∞ and $\|A\|_\infty = C$

(iii) If $C = \left(\sum_i \left(\sum_j |a_{ij}|^p \right)^{\frac{1}{q}} \right)^{\frac{1}{p}} < +\infty$ for $1 < p, q < +\infty, \frac{1}{p} + \frac{1}{q} = 1$ then A is a bounded linear operator on ℓ^p and $\|A\|_p \leq C$.

Pf: (i) Since $\sum_j |a_{ij} x_j| \leq \sup_j |a_{ij}| \sum_j |x_j|$
 $\sup_j |a_{ij}| \leq C < +\infty$
 $\sum_j |x_j| < +\infty$, since $\{x_j\} \in \ell^1$ } then $\sum_j |a_{ij} x_j| < +\infty$
 i.e. y_i is well-defined.

$$\|Ax\| = \|y_i\|_{\ell^1} = \sum_i |y_i| = \sum_i \left| \sum_j a_{ij} x_j \right| \leq \sum_i \sup_j |a_{ij}| \sum_j |x_j| \leq C \|x\|_{\ell^1}$$

Hence, A is bounded and $\|A\|_1 \leq C$.

It suffices to show $\|A\|_1 \geq C$, i.e. $\forall C' < C, \|A\|_1 > C'$.

Since $C = \sup_i \sum_j |a_{ij}| < +\infty$, then $\forall C' < C, \exists j_0$ s.t. $C' < \sum_j |a_{ij_0}|$

Let $e_j = (0, \dots, 0, \underset{j\text{-th}}{1}, 0, \dots)$ be a basis of ℓ^1

then $Ae_j = (a_{1j}, a_{2j}, \dots)$

$$C' < \sum_j |a_{ij_0}| = \|Ae_{j_0}\| \leq \|A\|_1 \|e_{j_0}\| = \|A\|_1$$

(ii) $\sum_j |a_{ij} x_j| \leq \sup_j |x_j| \sum_j |a_{ij}| \leq C \|x\|_{\ell^\infty} < +\infty, \forall x \in \ell^\infty$

So, y_i is well-defined and moreover

$$\|Ax\|_{\ell^\infty} = \|y\|_{\ell^\infty} = \sup_i |y_i| \leq \sup_j |x_j| \sup_i \sum_j |a_{ij}| \leq C \|x\|_{\ell^\infty}$$

Hence A is bounded and $\|A\|_\infty \leq C$

It suffices to show $\|A\|_\infty \geq C$, i.e. $\forall C' < C, \|A\|_\infty > C'$.

Since $\forall C' < C = \sup_i \sum_j |a_{ij}| < +\infty, \exists i_0$ s.t. $C' < \sum_j |a_{i_0 j}|$

Set $x_j = \text{sgn } a_{i_0 j}$, then $\|x\|_{\ell^\infty} = 1 < +\infty$ and

$$C' < \sum_j |a_{i_0 j}| = \sum_j a_{i_0 j} x_j = y_{i_0} \leq \|A\|_\infty \|x\|_{\ell^\infty} = \|A\|_\infty$$

(iii) By Hölder inequality

$$|y_i| \leq \left(\sum_j |a_{ij}|^q \right)^{\frac{1}{q}} \left(\sum_j |x_j|^p \right)^{\frac{1}{p}} < +\infty, \quad \forall x \in \ell^p.$$

Then A is well-defined

$$\|Ax\|_{\ell^p}^p = \|y\|_{\ell^p}^p = \sum_i |y_i|^p = \sum_i \left| \sum_j a_{ij} x_j \right|^p \leq \sum_i \left(\sum_j |a_{ij}|^q \right)^{\frac{p}{q}} \|x\|_{\ell^p}^p$$

$$\text{i.e. } \|y\|_{\ell^p} \leq \left(\sum_i \left(\sum_j |a_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \|x\|_{\ell^p}$$

Hence A is bounded and $\|A\| \leq C$.