MATH4010 Solution 4

P226. 5 Proof. $\forall x \in X$, $\exists N_n \in \mathbb{N}$, s.t. $\| x - \sum_{i=1}^{m} \alpha_i e_i \| < \frac{1}{n}$, $\forall m \geq N_n$.

Then $|||x||| \leq \sup_{1 \leq m \leq N_n} \| \sum_{i=1}^{M} \alpha_i e_i \| + \|x\| + \frac{1}{n}$. Therefore, $|||\cdot|||$ exists.

Since $\|x\| \leq \| \sum_{i=1}^{N_n} \alpha_i e_i \| + \frac{1}{n} \leq \sup_m \| \sum_{i=1}^{m} \alpha_i e_i \| + \frac{1}{n} \leq |||x||| + \frac{1}{n}$.

and n is arbitrary, $\|x\| \leq |||x|||$, $\forall x \in X$.

Then it's trivial to prove $|||\cdot|||$ is a complete norm.

So it follows from the Open Mapping Theorem that $\|\cdot\|$ and $|||\cdot|||$ are equivalent, i.e. $|||\cdot||| \leq c\|\cdot\|$ for some constant $c > 0$.

And thus $\|\phi_n\| \leq \frac{|\alpha_n|}{\|x\|} = \frac{\| \sum_{i=1}^{n} \alpha_i e_i - \sum_{i=1}^{n-1} \alpha_i e_i \|}{\|x\|} \leq \frac{2|||x|||}{\|x\|} \leq 2c$.

Since ϕ_n is obviously linear, $\phi_n \in X^*$. $\square$

Remark: $\{\phi_n\}_{n=1}^{\infty}$ may not be the Schauder basis for X^*. A counter example is that $L^1[0,1]$ has a Schauder basis, but $L^\infty[0,1)$ is not separable

6. Proof "$\Rightarrow$" Suppose that $M+N$ is closed. Since M, N are closed subspaces and $M \cap N = \{0\}$, then for any $x \in M+N$, $\exists$ unique $a \in M$, $b \in N$ s.t. $x = a+b$. Otherwise, $\exists\, a'(\neq a) \in M$, $b'(\neq b) \in N$, $x = a'+b'$, and then $a'-a = b-b'$ which contradicts to $M \cap N = \{0\}$.

Now define $\|\cdot\|'$ on $M+N$ as $\|x\|' = \|a\| + \|b\|$. The above arguments show that $\|\cdot\|'$ is well-defined. It is easy to prove $\|\cdot\|'$ is also a complete norm on $M+N$. And thus $\|\cdot\|$ and $\|\cdot\|'$ are equivalent by the Open Mapping Theorem.

$P: M+N \to M$ is obvious well-defined and linear.
$x+y \mapsto x$

Furthermore, $\|x\| \leq \|x\| + \|y\| = \|x+y\|' \leq c\|x+y\|$ for some $c>0$ independent of x.

Therefore, P is continuous.

"$\Leftarrow$" Suppose that P is continuous. Suppose that $\{x_n+y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X which converges to z in X, and $x_n+y_n \in M+N$.

So it suffices to prove $z \in M+N$.

Since P is continuous, $\{P(x_n+y_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Then $\exists\, x \in M$ s.t. $\lim_{n\to\infty}\|x-x_n\| = 0$. Similarily, $\exists\, y \in N$ s.t. $\lim_{n\to\infty}\|y-y_n\| = 0$. Therefore,

$$\|z-(x+y)\| \leq \|z-(x_n+y_n)\| + \|x_n - x\| + \|y_n - y\| \to 0$$

So $z = x+y \in M+N$. $\square$

9. Proof As in Exercise 6, the norm $\|\cdot\|'$ on $M+N$ satisfies $\|\cdot\|' \leq c\|\cdot\|$ for some $c>0$. Therefore, for any unit vectors $u \in M$, $v \in N$, we have

$$\|u-v\| \geq \frac{1}{c}\|u-v\|' = \frac{1}{c}\left(\|u\| + \|-v\|\right) = \frac{2}{c}.$$

$\square$