MATH Solution 2

3. Proof. It is trivial that R is a linear operator, and thus it suffices to prove that R is bounded.

Actually, $\|Rx\|_{\ell^\infty} = \|x\|_{\ell^\infty} \leq \|x\|_{\ell^1}$

and then $\|R\| \leq 1$. $\square$

4. Proof. $T: \ell^1 \to \ell^1$ is linear and bounded, and thus it's also continuous. $\ker T = \{0\}$, and therefore T is 1-1.

Suppose that $x_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, \ldots)$

$Tx_n = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, 0, 0, \ldots)$

Then Tx_n converges to $y = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \frac{1}{(n+1)^2}, \ldots) \in \ell^1$ in ℓ^1 norm. But $y \notin \ell^1_1$.

Actually, if x satisfies $Tx = y$, then $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$ but $x \notin \ell^1$. So the image of T is not closed. $\square$

5. Proof. It is easy to prove that D is linear.

$DT = Id_{\ell^1}$, $TD = Id_{\ell^1_1}$, here Id_X denotes the identity map from X to X. Therefore D is invertible.

To prove D is not continuous, it suffices to prove D is not bounded, i.e., prove that there exist a sequence $x_n \in \ell^1_1$, $C \in \mathbb{R}^+$ such that $\lim_{n\to\infty} \|x_n\|_{\ell^1} = 0$, $\|Dx_n\|_{\ell^1} \leq C$. Actually, $x_n = \frac{e_n}{n}$, $C = 1$. $\square$

13. <u>Proof</u> The linearity of the integral operation implies the linearity of T.

So it suffices to prove $\|Tf\|_{L^\infty[1,\infty)} \leq \|f\|_{L^\infty[1,\infty)}$

$$
\begin{aligned}
\|Tf\|_{L^\infty} &= \left\| \int_1^\infty x^{-(y+1)} f(x)\,dx \right\|_{L^\infty[1,\infty)} \\
&\leq \left\| \|f\|_{L^\infty[1,\infty)} \int_1^\infty x^{-(y+1)}\,dx \right\|_{L^\infty[1,\infty)} \\
&= \|f\|_{L^\infty[1,\infty)} \left\| \int_1^\infty x^{-(y+1)}\,dx \right\|_{L^\infty[1,\infty)} \\
&= \|f\|_{L^\infty[1,\infty)} \left\| \frac{1}{y} \right\|_{L^\infty[1,\infty)} = \|f\|_{L^\infty[1,\infty)}.
\end{aligned}
$$

$\square$