Real Analysis

Tutorial 8 (March 15, 17)

The following were discussed in the tutorial this week.

1. Prove that if f is integrable on $\mathbb{R}$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e. x. As a result, if $\int_E f(x)dx = 0$ for every measurable E, then $f = 0$ a.e.

2. (Chebyhev's inequality) Let $E \subseteq \mathbb{R}$ be measurable, and let f be a measurable function on E. Then for all $\alpha > 0$,
$$m\{x \in E : |f(x)| \geq \alpha\} \leq \frac{1}{\alpha} \int_E |f|.$$

3. Let f be an integrable function on E such that $\int_E |f| = 0$. Show that $f = 0$ a.e. on E.

4. Find the limit
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{(\sin x)^n}{x^2} dx.$$

5. Discuss the following modes of convergence for sequences of measurable functions. Let $E \subseteq \mathbb{R}$ be measurable. Let f, f_n be measurable functions on E.

   (a) Almost everywhere convergence: $f_n \to f$ a.e.,
   that is, there exists $A \subseteq E$ wit $m(E \setminus A) = 0$ such that $f_n(x) \to f(x)$ for all $x \in A$.

   (b) L^1-convergence: $f_n \to f$ in L^1, that is, $\|f_n - f\|_1 := \int_E |f_n - f| \to 0$.

   (c) Convergence in measure: $f_n \to f$ in measure,
   that is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon$.

   These modes of convergence satisfy the following relations:

   (i) $f_n \to f$ in L^1 $\Rightarrow$ there is a subsequence $f_{n_k} \to f$ a.e.

   (ii) $f_n \to f$ in measure $\Longleftrightarrow$ every subsequence of $\{f_n\}$ has a subsequence that converges to f a.e. The converse hols if we further assume that $m(E) < \infty$.

   (iii) $f_n \to f$ in L^1 $\Rightarrow$ $f_n \to f$ in measure.