# MATH 4050 Real Analysis

## Tutorial 11 (April 5, 7)

The following were discussed in the tutorial this week.

1. Let $E_n$ be an increasing sequence of subsets of $\mathbb{R}$ (not necessarily measurable). Set $E = \bigcup_{n=1}^{\infty} E_n$. Show that
$$\lim_n m^*(E_n) = m^*(E),$$
where $m^*$ is the (Lebesgue) outer measure.

   **Remark:** If each $E_n$ is measurable, the result is already known. This result and the outer regularity of $m^*$ can be used to prove the limit above.

2. We prove the change of variable formula for Lebesgue integral in several steps. Let $g : [a,b] \to [c,d]$ be a monotone increasing absolutely continuous function such that $g(a) = c$ and $g(b) = d$.

   (a) Show that for any $G_\delta$ set $G \subseteq [c,d]$,
$$m(G) = \int_{g^{-1}(G)} g'(x)dx.$$

   (b) Let $H = \{x \in [a,b] : g'(x) \neq 0\}$. If $E \subseteq [c,d]$ has measure zero, show that $g^{-1}(E) \cap H$ has measure zero.

   (c) If $E \subseteq [c,d]$ is measurable, show that $F := g^{-1}(E) \cap H$ is measurable and
$$m(E) = \int_F g' = \int_a^b \chi_E(g(x))g'(x)dx.$$
(Note that $g^{-1}(E)$ and hence $\chi_E(g(x))$ may not be measurable.)

   (d) If $f$ is a non-negative measurable function on $[c,d]$, show that $(f \circ g)g'$ is measurable on $[a,b]$ and
$$\int_c^d f(y)dy = \int_a^b f(g(x))g'(x)dx.$$
Prove the corresponding result where $f$ is integrable.

3. A function $f : [a,b] \to \mathbb{R}$ is said to be *singular* if for almost every $x \in [a,b]$, $f'(x)$ exists and is equal to 0.

   **Example:** Cantor function

4. Show that if $f$ is both absolutely continuous and singular on $[a,b]$, then $f$ is a constant on $[a,b]$.

   (**Hint:** Use Luzin N property and (2) in tutorial 10.)

5. Lebesgue decomposition theorem:

   (a) If $f$ is an increasing function on $[a,b]$, then there exist an absolutely continuous increasing function $g$ and a singular increasing function $h$ on $[a,b]$ such that $f = g + h$. Moreover the decomposition is unique up to constants.

   (b) If $f$ is a function of bounded variation on $[a,b]$, then there exist an absolutely continuous function $g$ and a singular function of bounded variation $h$ on $[a,b]$ such that $f = g + h$. Moreover the decomposition is unique up to constants.