Assignment 6

1. Let E, A be measurable with $A \subseteq E$, and let $f : E \to [-\infty, \infty]$ be measurable. With respect to each of the sections in Ch. 4 we have discussed $\int_E f$ (and hence $\int_A f$ by letting $E = A$); another approach is to let $\int_A f = \int_E f \chi_A$. Show that these two approaches are equivalent (in each of the four sections). Also, for each of the four sections, show that if $f \sim \tilde{f}$ ($f(x) = \tilde{f}(x)$ for almost every x) then $\int_E f = \int_E \tilde{f}$.

2. Let $f \in \mathcal{L}_1(E)$. Show that $A \mapsto \int_A f$ is countably additive over the measurable subsets A of E in each of the following lines:
   (a) Apply the MCT to the special case when $f \geq 0$.
   (b) Without using MCT, direct go to the general case by appplying the Lebesgue Dominated Convergence Th.

3. Provide a counter-example for each of the following:
   (i) Replace $0 \leq f_n$ in Fatou by $-1 \leq f_n$;
   (ii) Replace $f_n \uparrow$ in MCT by $f_n \downarrow$;
   (iii) Drop the boundedness assumption in the Bounded Conv. Th.
   (iv) Drop the finiteness assumption $m(E) < +\infty$ in - - -

4. Let $f \in \mathcal{L}_1(E)$ with $m(E) \leq +\infty$. Show that $f(x) \in \mathbb{R}$ for a.e. $x \in E$, and that $\int_E |f_n - f| \to 0$ as $n \to +\infty$, where $f_n = (-n) \vee f \wedge n$, i.e.

$$f_n(x) = \begin{cases} -n & \text{if } f(x) < -n \\ f(x) & \text{if } -n \leq f(x) \leq n \\ n & \text{if } f(x) > -n \end{cases}.$$

(thus $|f_n - f| \leq 2|f|$ a.e on E). Show further that, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\int_A |f| < \varepsilon$$

whenever $A \subseteq E$ with $m(A) < \delta$.

Hint: Take $N \in \mathbb{N}$ s.t. $\int_E |f_N - f| < \varepsilon/2$, and take $\delta = \frac{\varepsilon}{2N}$. Then, if $m(A) < \delta$, one has (?)

$$\int_A |f| \leq \int_A (|f - f_N| + |f_N|) \leq \int_A |f - f_N| + N \cdot m(A) < \varepsilon.$$