THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Extra Tutorial 1 (November 2)

Definition (Open interval cover)**.** *Let B be a subset of $\mathbb{R}$. A collection $\mathcal{C} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of subsets of $\mathbb{R}$ is said to be an **open interval cover** of B if*

(a) U_α is an open interval for all $\alpha \in \mathcal{A}$;

(b) $B \subseteq \bigcup_{\alpha\in\mathcal{A}} U_\alpha$.

Definition (Compact set)**.** *A subset $B \subseteq \mathbb{R}$ is said to be **compact** if every open interval cover $\mathcal{C} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of B has a finite subcover, that is*

$$B \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_k} \quad \text{for some } \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathcal{A}.$$

Example. *1. $\mathcal{C} = \left\{\left(0, 1 - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is an open interval cover of $(0,1)$ with no finite subcover.*

2. $\mathcal{C} = \left\{\left(-\frac{1}{n}, 1 + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ is an open interval cover of $[0,1]$ with a finite subcover.

3. $\mathcal{C} = \{(-n, n) : n \in \mathbb{N}\}$ is an open interval cover of $\mathbb{R}$ with no finite subcover.

Theorem (Heine-Borel Theorem)**.** *Every closed and bounded interval $[a,b]$ is compact.*

Follow the exercises below to give two proofs of Heine-Borel Theorem.

Exercise 1. *Assume $a < b$. Let $\mathcal{C}$ be an open interval cover of $[a,b]$. Define*

$$S := \{x \in [a,b] : [a,x] \text{ is covered by a finite subcollection of } \mathcal{C}\}.$$

(a) Show that S is non-empty and bounded.

(b) Let $u = \sup S$. Show that $u \in S$.

(c) Show that $u = b$.

Exercise 2. *Without loss of generality, assume that $[a,b] = [0,1] =: I$. Let $\mathcal{C}$ be an open interval cover of I. Suppose $\mathcal{C}$ has no finite subcover for I.*

(a) Show that there is a sequence $\{I_n\}$ of closed bounded intervals such that

- *(i) $I \supset I_1 \supset I_2 \supset \cdots$*
- *(ii) Each I_n is not covered by any finite subcollection of $\mathcal{C}$.*
- *(iii) If $x, y \in I_n$, then $|x - y| \leq 2^{-n}$.*

(b) Deduce a contradiction.