THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 5 (October 10)

The following were discussed in the tutorial this week:

(I am sorry that I made some mistakes in 2 and 3 in the tutorials. Please check the remarks below.)

1. Let (a_n) be a bounded sequence of real numbers. For each $n \in \mathbb{N}$, define

$$b_n = \sup_{k \geq n} a_k = \sup\{a_k : k \geq n\}, \qquad c_n = \inf_{k \geq n} a_k = \inf\{a_k : k \geq n\}.$$

   (a) Show that (b_n) and (c_n) are monotone, and hence convergent. Moreover,

$$\lim_n b_n = \inf_n b_n, \quad \text{and} \quad \lim_n c_n = \sup_n c_n.$$

   (b) The limits $\lim_n b_n$ and $\lim_n c_n$ are called the **limit superior** and **limit inferior** of (a_n), respectively. That is,

$$\overline{\lim_n}\, a_n := \lim_n b_n = \inf_{n \geq 1} \left(\sup_{k \geq n} a_n \right), \qquad \underline{\lim_n}\, a_n := \lim_n c_n = \sup_{n \geq 1} \left(\inf_{k \geq n} a_n \right).$$

   Show that $\underline{\lim_n}\, a_n \leq \overline{\lim_n}\, a_n$.

   (c) Show that (a_n) converges to ℓ if and only if $\overline{\lim_n}\, a_n = \underline{\lim_n}\, a_n = \ell$.

Remark: If (a_n) is unbounded above (below), define $\overline{\lim_n}\, a_n = +\infty$ ($\underline{\lim_n}\, a_n = -\infty$).

2. Let (a_n) be a bounded sequence of real numbers. Show that

   (a) (i) if $\overline{\lim_n}\, a_n < \alpha$, then there is $N \in \mathbb{N}$ such that $a_n < \alpha$ for all $n \geq N$.

   (ii) if $\overline{\lim_n}\, a_n > \alpha$, then for all $n \in \mathbb{N}$, there is $k \geq n$ such that $a_k > \alpha$.

   (b) (i) if $\underline{\lim_n}\, a_n > \beta$, then there is $N \in \mathbb{N}$ such that $a_n > \beta$ for all $n \geq N$.

   (ii) if $\underline{\lim_n}\, a_n < \beta$, then for all $n \in \mathbb{N}$, there is $k \geq n$ such that $a_k < \beta$.

Remark: I am sorry that I made a mistake in tutorials claiming that the converse of the above statements were also true. This is not correct.

3. Let (a_n) be a bounded sequence of real numbers. Define

$$E := \{x \in \mathbb{R} : \text{there is a subsequence } (a_{n_k}) \text{ of } (a_n) \text{ such that } a_{n_k} \to x\}.$$

Let $\alpha = \overline{\lim_n} a_n$. Show that $\alpha \in E$ and $\alpha = \sup E$. An analogous result holds for limit inferior.

Remark: I am sorry that I made a mistake in the proof of $\alpha \in E$ in the first tutorial. A correct proof is given below:

By 2(a)(ii), there is $n_1 \in \mathbb{N}$ such that $\alpha - \frac{1}{2^1} < a_{n_1}$.

By 2(a)(ii), there is $n_2 \geq n_1 + 1$ such that $\alpha - \frac{1}{2^2} < a_{n_2}$.

By 2(a)(ii), there is $n_3 \geq n_2 + 1$ such that $\alpha - \frac{1}{2^3} < a_{n_3}$.

Continue in this way, we find $n_1 < n_2 < n_3 < \cdots$ such that for all $k \in \mathbb{N}$,

$$\alpha - \frac{1}{2^k} < a_{n_k} \leq b_{n_k} := \sup\{a_m : m \geq n_k\}.$$

Since $\lim_k \left(\alpha - \frac{1}{2^k}\right) = \lim_k b_{n_k} = \alpha$, it follows from the squeeze theorem that $\lim_k a_{n_k} = \alpha$. Since (a_{n_k}) is a subsequence of (a_n), we have $\alpha \in E$ $\square$.

4. Let (x_n) and (y_n) be bounded sequences of real numbers. Show that

   (a) $\overline{\lim_n}(-x_n) = -\underline{\lim_n}\, x_n$;

   (b) if $x_n \leq y_n$ for all n, then $\overline{\lim_n}\, x_n \leq \overline{\lim_n}\, y_n$ and $\underline{\lim_n}\, x_n \leq \underline{\lim_n}\, y_n$;

   (c) $\underline{\lim_n}\, x_n + \underline{\lim_n}\, y_n \leq \underline{\lim_n}(x_n + y_n)$ and $\overline{\lim_n}(x_n + y_n) \leq \overline{\lim_n}\, x_n + \overline{\lim_n}\, y_n$.

5. Let (x_n) be a sequence of positive real numbers. Show that

$$\underline{\lim_n} \frac{x_{n+1}}{x_n} \leq \underline{\lim_n} \sqrt[n]{x_n} \leq \overline{\lim_n} \sqrt[n]{x_n} \leq \overline{\lim_n} \frac{x_{n+1}}{x_n}.$$

6. Let (x_n) be a sequence of real numbers. Show that

$$\underline{\lim_n}\, x_n \leq \underline{\lim_n} \frac{x_1 + \cdots + x_n}{n} \leq \overline{\lim_n} \frac{x_1 + \cdots + x_n}{n} \leq \overline{\lim_n}\, x_n.$$