THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 11 (November 21)

The following were discussed in the tutorial this week:

1. Let f and g be uniformly continuous on $A \subseteq \mathbb{R}$. If f, g are both bounded on A, show that fg is uniformly continuous on A. Can the boundedness condition be dropped?

2. Let $f(x) := x$ and $g(x) = \sin x$. Show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

3. Let $f : A \to \mathbb{R}$ be a function.
   (a) If f is uniformly continuous, show that f maps any Cauchy sequence to a Cauchy sequence.
   (b) Is the converse true?
   (c) Prove the converse if we further assume that A is bounded.

4. Suppose $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$ and $\lim_n f(n + h) = L$ for any $h \in [0, 1]$. Show that $\lim_{x\to\infty} f(x) = L$.

5. Let A be a nonempty subset of $\mathbb{R}$. For $x \in \mathbb{R}$, define
   $$\rho_(x) = \inf\{|x - y| : y \in A\}.$$
   (a) Show that ρ_A is uniformly continuous on A.
   (b) Show that, for any $x \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that
   $$\rho_A(x) = |x - z|.$$
   Does z always belong to A?