Selected solution to 2050B Test 1

2. (a). Firstly, note that

$$\left|\frac{1}{y_n} - 4\right| = \left|\frac{1 - 4y_n}{y_n}\right| = \frac{4 \cdot \left|\frac{1}{4} - y_n\right|}{|y_n|}.$$

Let $\varepsilon > 0$. We want to show that there exists an $N_\varepsilon \in \mathbb{N}$ such that whenever $n > N_\varepsilon$, the value of this expression is less than ε.*

Since $\lim y_n = 1/4$, by using the definition of lim, there exists $N_1 \in \mathbb{N}$ such that whenever $n > N_1$,

$$|y_n - 1/4| < 1/8.$$

Therefore, whenever $n > N_1$,

$$-1/8 < y_n - 1/4 < 1/8,$$

or

$$1/8 < y_n < 3/8.$$

This means whenever $n > N_1$, y_n is positive, and $|y_n| = y_n > 1/8$. Next, let $\varepsilon' := (1/8) \cdot (1/4) \cdot \varepsilon$, which is a positive number. By using the definition of $\lim y_n = 1/4$ again, there exists $N_{\varepsilon'} \in \mathbb{N}$ such that whenever $n > N_{\varepsilon'}$,

$$|y_n - 1/4| < \varepsilon'.$$

As a result, let $N_\varepsilon := \max(N_1, N_{\varepsilon'})$, then whenever $n > N_\varepsilon$, we

*Since $\lim y_n = 1/4$, the appearance of $\left|\frac{1}{4} - y_n\right|$ helps to bring it small. It remains to control $\frac{4}{|y_n|}$.

have

$$\begin{aligned}\left|\frac{1}{y_n}-4\right| &= \frac{4\cdot\left|\frac{1}{4}-y_n\right|}{|y_n|}\\ &< \frac{4\cdot\left|\frac{1}{4}-y_n\right|}{1/8}\quad\text{(since } n>N_1\text{)}\\ &< \frac{4\cdot\varepsilon'}{1/8}\quad\text{(since } n>N_{\varepsilon'}\text{)}\\ &= \varepsilon.\quad\text{(by the definition of } \varepsilon'\text{)}\end{aligned}$$

By (ε-N) terminology, we conclude that $(1/y_n)$ converges to 4.

(b). Firstly, observe that

$$\begin{aligned}\left|\frac{z_n^3-3}{z_n^2-3}-5\right| &= \left|\frac{z_n^3-5z_n^2+12}{z_n^2-3}\right|\\ &= \left|\frac{(z_n-2)(z_n^2-3z_n-6)}{z_n^2-3}\right|\\ &= \frac{|z_n-2|\cdot|z_n^2-3z_n-6|}{|z_n^2-3|}.\end{aligned}$$

(see footnote for a thought of the second equality†)

Let $\varepsilon>0$. We want to show that there exists an $N_\varepsilon\in\mathbb{N}$ such that whenever $n>N_\varepsilon$, the value of this expression is less than ε.‡

Since $\lim z_n=2$, by using the definition of lim, there exists $N_1\in\mathbb{N}$ such that whenever $n>N_1$,

$$|z_n-2|<0.1.$$

Therefore, whenever $n>N_1$,

$$-0.1<z_n-2<0.1,$$

†We expect that this quantity is close to zero when z_n is close to 2 (so that the question gives a true statement). When $z_n\approx 2$, the denominator is $\approx 2^2-3=1$, so to force this fraction close to zero , the numerator should be close to zero. As a polynomial, this means $(x-2)$ should be a factor of (x^3-5x^2+12).

‡Since $\lim z_n=2$, the appearance of $|z_n-2|$ helps to bring it small. It remains to control $\left|z_n^2-3z_n-6\right|/\left|z_n^2-3\right|$.

or

$$1.9 < z_n < 2.1,$$

whence

$$3.61 = (1.9)^2 < z_n^2 < (2.1)^2 = 4.41.$$

As $3.61 - 3 = 0.61 > 0$, therefore whenever $n > N_1$ we have $|z_n^2 - 3| = z_n^2 - 3 > 0.61$.

On the other hand, observe that for all $x \in \mathbb{R}$, $|x^2 - 3x - 6| \leq |x|^2 + |3| \cdot |x| + |6|$ by triangle inequality.[§] Therefore, whenever $n > N_1$,

$$\begin{aligned} \left|z_n^2 - 3z_n - 6\right| &\leq |z_n|^2 + |3| \cdot |z_n| + |6| \\ &< (2.1)^2 + 3 \cdot (2.1) + 6 \\ &< 3^2 + 3 \cdot 3 + 6 \\ &= 24. \end{aligned}$$

Next, let $\varepsilon' := (0.61) \cdot (1/24) \cdot \varepsilon$, which is a positive number. By using the definition of $\lim z_n = 2$ again, there exists $N_{\varepsilon'} \in \mathbb{N}$ such that whenever $n > N_{\varepsilon'}$,

$$|z_n - 2| < \varepsilon'.$$

As a result, let $N_\varepsilon := \max(N_1, N_{\varepsilon'})$, then whenever $n > N_\varepsilon$, we have

$$\begin{aligned} \left|\frac{z_n^3 - 3}{z_n^2 - 3} - 5\right| &= \frac{|z_n - 2| \cdot |z_n^2 - 3z_n - 6|}{|z_n^2 - 3|} \\ &< \frac{|z_n - 2| \cdot |z_n^2 - 3z_n - 6|}{0.61} \quad (\text{since } n > N_1) \\ &< \frac{|z_n - 2| \cdot 24}{0.61} \quad (\text{since } n > N_1) \\ &< \frac{\varepsilon' \cdot 24}{0.61} \quad (\text{since } n > N_{\varepsilon'}) \\ &= \varepsilon. \quad (\text{by the definition of } \varepsilon') \end{aligned}$$

[§] Some mathematicians call this technique "trivial bound", because it ignores the possible cancellation inside the absolute value. For example, $\left|4^2 - 2 \cdot 5 - 6\right| = 0$, while $4^2 + 2 \cdot 5 + 6 = 32$ only gives a bad estimate. This technique is indeed useful. An advantage is that it is easy to obtain, so as long as we have another term to compensate the rough estimation we are happy with it.

By (ε-N) terminology, we conclude that $(\frac{z_n^3-3}{z_n^2-3})$ converges to 5.

4. A sequence (a_n) is Cauchy iff $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall m > N_\varepsilon$, $\forall n > N_\varepsilon$, we have $|a_n - a_m| < \varepsilon$.

   A sequence (a_n) is not Cauchy iff $\exists \varepsilon_0 > 0$ such that $\forall N \in \mathbb{N}$, $\exists m > N$, $\exists n > N$, such that $|a_n - a_m| \geq \varepsilon_0$.