MATH4050 Real Analysis

Assignment 2

Only star-questions will be marked.

The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$. $x \in \mathbb{R}$ is called a point of closure of E if each neighbourhood of x intersects E.

1.* (3rd: P.39, Q12)
Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x.
Q: How about $x \in \{-\infty, \infty\}$?

2. (3rd: P.39, Q13)
Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists k \geq n$ such that $x_k > l - \varepsilon$.

3.* (3rd: P.39, Q14)
Show that $\limsup x_n = \infty$ if and only if given Δ and n, $\exists k \geq n$ such that $x_k > \Delta$.

4. (3rd: P.39, Q15)
Show that $\liminf x_n \leq \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

5.* (3rd: P.39, Q16)
Prove that

$$\limsup x_n + \liminf y_n \leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

provided the right and left sides are not of the form $\infty - \infty$.

6. (3rd: P.39, Q17)
Prove that if $x_n > 0$ and $y_n \geq 0$, then

$$\limsup (x_n y_n) \leq (\limsup x_n)(\limsup y_n)$$

provided the product on the right is not of the form $0 \cdot \infty$.

7. (3rd: P.46, Q27)
Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))
A number x is called an **accumulation point** of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))
Show that $\overline{E} = E \cup E'$.

10. (3rd: P.46, Q30; 4th: P.20, Q31)
A set E is called **isolated** if $E \cap E' = \phi$. Show that every isolated set of real numbers is countable.

11.* Let $f : [0,1] \to [m, M]$ with the upper Riemann integral $\alpha = (R)\overline{\int_0^1} f(x)\,dx$. Show: $\exists$ a sequence (ψ_n) of step-functions such that $\int_0^1 \psi_n(x)\,dx \to \alpha$ and $\psi_n(x) \downarrow \bar{f}(x)$ $\forall x \in [0,1] \setminus \left\{\frac{k}{2^n} : n \in \mathbb{N}, k = 0, 1, \cdots, 2^n\right\}$, where $\bar{f}$ is defined as below in Q 11(a).

11(a) Let $f : [0,1] \to [m,M]$ be Riemann integrable. Find a sequence (ψ_n) of step-functions such that

$$\psi_n(x) \downarrow \bar{f}(x) \quad \forall x \in \underbrace{[0,1] \setminus \left\{ \frac{k}{2^n} : n \in \mathbb{N},\ k = 0, \dots, 2^n \right\}}_{\text{to be denoted by } X}$$

and

$$\int_0^1 \psi_n(x)\,dx \to \int_0^1 f(x)\,dx,$$

where

$$\bar{f}(x) \stackrel{\text{def}}{=} \inf\{ f^\delta(x) : \delta > 0 \}, \quad \forall x \in [0,1]$$

with each

$$f^\delta(x) = \sup\{ f(u) : u \in V_\delta(x) \cap [0,1] \}, \quad \forall x.$$

Hint: Partition $[0,1]$ into 2^n-many subintervals of equal length ($= 1/2^n$) and get your ψ_n correspondingly in light of upper Riemann sum.

Let $x_0 \in X$ and suppose x_0 lies in I, one of the above subintervals. Then x_0 must be in the interior of I so $\exists\, \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$, then $f^\delta(x_0) \le \psi_n(x_0)$ so $\bar{f}(x_0) \le \psi_n(x_0)$ $\forall n$. Conversely let $x_0 \in X$ and $\delta > 0$. Take n such that $1/2^n < \delta$. If I is … then $x_0 \in I \subseteq (x_0 - \delta, x_0 + \delta)$ as the length of I is smaller than δ. Then $\psi_n(x_0) \le f^\delta(x_0)$ so $\inf\{ \psi_k(x_0) : k \in \mathbb{N} \} \le \psi_n(x_0) \le f^\delta(x_0)$ $\forall \delta > 0$.