Midterm Test for MATH4220

March 9, 2017

1. (**20 points**)

   (a) (**10 points**) Find all the solutions to

   $$\partial_x u - 2\partial_y u + 2u = 1$$

   (b) (**10 points**) Solve the problem

   $$\begin{cases} y\partial_x u + 3x^2 y\,\partial_y u = 0 \\ u(x=0,y) = y^2 \end{cases}$$

   In which region of the xy-plane is the solution uniquely determined?

2. (**20 points**)

   (a) (**4 points**) What is the type of the equation $\partial_t^2 u + \partial_{xt}^2 u - 2\partial_x^2 u = 0$?

   (b) (**16 points**) Solve the Cauchy problem

   $$\begin{cases} \partial_t^2 u + \partial_{xt}^2 u - 2\partial_x^2 u = 2, & -\infty < x < +\infty, \quad -\infty < t < +\infty \\ u(x,t=0) = x^2, \quad \partial_t u(x,t=0) = 0 \end{cases}$$

3. (**20 points**)

   (a) (**5 points**) State the definition of a well-posed PDE problem.

   (b) (**5 points**) Is the following boundary value problem well-posed? Why?

   $$\begin{cases} \dfrac{d^2u}{dx^2} + \dfrac{du}{dx} = 1, & 0 < x < 1 \\ u'(0) = 1, \quad u'(1) = 0 \end{cases}$$

   (c) (**10 points**) State and prove the uniqueness and continuous dependence of solutions to the problem

   $$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < 1, \quad 0 < t < T, \quad T > 0 \\ \partial_x u(0,t) = 0, \quad \partial_x u(1,t) = 0, & t > 0 \\ u(x,t=0) = \varphi(x) \end{cases}$$

4. (**20 points**)

(a) (**15 points**) Derive the formal solution formula to the problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < +\infty, \quad t > 0 \\ \partial_x u(x = 0, t) = 0, & t > 0 \\ u(x, t = 0) = \varphi(x), & 0 < x < +\infty \end{cases}$$

by the method of reflection (with all the details of the derivation).

(b) (**5 points**) Let $\varphi(x) = \cos x$, $0 < x < +\infty$. Find the maximum value of $u(x, t)$.

5. (**20 points**)

(a) (**10 points**) Prove the following generalized maximum principle:
If $\partial_t u - k\partial_x^2 u \leq 0$ on $R \triangleq [0, l] \times [0, T]$ with a positive constant k, then

$$\max_R u(x, t) = \max_{\partial R} u(x, t)$$

here $\partial R = \{(x, t) \in R |$ either $t = 0$, or $x = 0$, or $x = l\}$.

(b) (**10 points**) Show if $v(x, t)$ solves the following problem

$$\begin{cases} \partial_t v = k\partial_x^2 v + f(x, t), & 0 < x < l, \quad 0 < t < T \\ v(x, 0) = 0, & 0 < x < l \\ v(0, t) = 0, \quad v(l, t) = 0, & 0 \leq t \leq T \end{cases}$$

with a continuous function f on $R \triangleq [0, l] \times [0, T]$.
Then,

$$v(x, t) \leq t \max_R |f(x, t)|$$

(Hint, consider $u(x, t) = v(x, t) - t \max_R |f(x, t)|$ and apply the result in (a).)