TA's solution to 2060B homework 6

p.215 Q12. (3 marks)

We need to show that $\forall \varepsilon > 0$, there exists a partition P_ε on $[0,1]$ s.t $U(g, P_\varepsilon) - L(g, P_\varepsilon) < \varepsilon$.

Let $0 < \varepsilon < 4$. Since g is continuous on $[\varepsilon/4, 1]$, we have $g|_{[\varepsilon/4,1]} \in \mathcal{R}[\varepsilon/4, 1]$. Hence, there exists a partition Q_ε on $[\varepsilon/4, 1]$ s.t $U(g|_{[\varepsilon/4,1]}, Q_\varepsilon) - L(g|_{[\varepsilon/4,1]}, Q_\varepsilon) < \varepsilon/2$. Define the partition P_ε on $[0,1]$ by prepending 0 to Q_ε, so that

$$U(g, P_\varepsilon) = \sup(g\left[0, \frac{\varepsilon}{4}\right]) \cdot \frac{\varepsilon}{4} + U(g|_{[\varepsilon/4,1]}, Q_\varepsilon),$$

$$L(g, P_\varepsilon) = \inf(g\left[0, \frac{\varepsilon}{4}\right]) \cdot \frac{\varepsilon}{4} + L(g|_{[\varepsilon/4,1]}, Q_\varepsilon).$$

Noting that $|g| \leq 1$, we have

$$\begin{aligned} U(g, P_\varepsilon) - L(g, P_\varepsilon) &\leq 2 \cdot \frac{\varepsilon}{4} + U(g|_{[\varepsilon/4,1]}, Q_\varepsilon) - L(g|_{[\varepsilon/4,1]}, Q_\varepsilon) \\ &< 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which was to be demonstrated.

p.215 Q18. (4 marks)

Plainly we have the result if $f \equiv 0$. Else, $\exists x_0 \in [a,b]$ s.t.

$$f(x_0) = \sup(f[a,b]) > 0.$$

Let $0 < \varepsilon < f(x_0)$. By the continuity of f, $\exists \delta > 0$ s.t. $f(x) > f(x_0) - \varepsilon$ $\forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b] := I$. Note that I is always an interval of positive length, regardless of whether $x_0 \in (a,b)$ or $x_0 \in \{a,b\}$. Denote its length by ℓ_ε.*

Consider the step function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x_0) - \varepsilon & \text{if } x \in I \\ 0 & \text{otherwise.} \end{cases}$$

*Note that $\ell_\varepsilon < \delta$ if $\delta = 100(b-a)$.

Note that $\forall n \in \mathbb{N}$ we have $g^n \leq f^n$,[†] and that $f^n \leq f(x_0)^n$. Therefore,

$$(f(x_0) - \varepsilon)^n \ell_\varepsilon \leq \int_a^b f^n \leq f(x_0)^n \cdot (b-a),$$

whence

$$(f(x_0) - \varepsilon)\ell_\varepsilon^{\frac{1}{n}} \leq M_n \leq f(x_0) \cdot (b-a)^{\frac{1}{n}}.$$

The proof is then completed by one of the following argument:

(a) We use the tool of $\overline{\lim}$ and $\underline{\lim}$.[‡§]
Since $\forall n \in \mathbb{N}$ we have

$$(f(x_0) - \varepsilon)\ell_\varepsilon^{\frac{1}{n}} \leq M_n \leq f(x_0) \cdot (b-a)^{\frac{1}{n}},$$

therefore

$$f(x_0) - \varepsilon \leq \underline{\lim} M_n \quad \text{and} \quad \overline{\lim} M_n \leq f(x_0).$$

By considering $\varepsilon \downarrow 0$, we have $f(x_0) \leq \underline{\lim} M_n$, whence $\underline{\lim} M_n = \overline{\lim} M_n = f(x_0)$, so $\lim M_n$ exists and $\lim M_n = f(x_0)$.

(b) We use $\varepsilon - N$ criterion.[¶]
Since both $\ell_\varepsilon^{\frac{1}{n}}$ and $(b-a)^{\frac{1}{n}}$ tend to 1 when $n \to \infty$, so $\exists N$ s.t. $\forall n \geq N$,

$$1 - \varepsilon \leq \ell_\varepsilon^{\frac{1}{n}} \quad \text{and} \quad (b-a)^{\frac{1}{n}} \leq 1 + \varepsilon.$$

It follows that $\forall n \geq N$,

$$f(x_0) - f(x_0)\varepsilon - \varepsilon + \varepsilon^2 = (f(x_0) - \varepsilon)(1 - \varepsilon) \leq M_n \leq f(x_0)(1 + \varepsilon),$$

whence

$$f(x_0) - (f(x_0) + 1)\varepsilon \leq M_n \leq f(x_0) + (f(x_0) + 1)\varepsilon.$$

Thus, $\forall n \geq N$ we have $|M_n - f(x_0)| \leq (f(x_0) + 1)\varepsilon$. The result follows.

[†] We need $f \not\equiv 0$ here (consider $n = 2$ otherwise).

[‡] For their meaning please refer to textbook Section 3.4 or 2050B Hw5.

[§] At this moment we still do not know if $\lim M_n$ exists or not. Therefore, we cannot just let $n \to \infty$ and write $f(x_0) - \varepsilon \leq \lim M_n \leq f(x_0)$. Consider e.g the sequence $\{(-1)^n\}$ and the inequality $-1 \leq (-1)^n \leq 1$. Nevertheless, since $\overline{\lim}$ and $\underline{\lim}$ are defined as the limits of some monotone sequences, so they always exist in $\mathbb{R} \cup \{-\infty, \infty\}$. Their values and difference helps us to study the tail behavior of a sequence.

[¶] A student provides this solution.

(c) This is essentially the same approach of using $\overline{\lim}$ and $\underline{\lim}$, but we can assume unfamiliarity with them.

Since (M_n) is a bounded sequence, by Bolzano-Weierstrass Theorem it has a convergent subsequence (M_{n_k}).

As $\forall n \in \mathbb{N}$ we have

$$(f(x_0) - \varepsilon)\ell_\varepsilon^{\frac{1}{n}} \leq M_n \leq f(x_0) \cdot (b-a)^{\frac{1}{n}},$$

therefore

$$f(x_0) - \varepsilon \leq \lim_{k\to\infty} M_{n_k} \leq f(x_0).$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{k\to\infty} M_{n_k} = f(x_0)$. This shows that every convergent subsequence of (M_n) converges to $f(x_0)$. Therefore it must be that $\lim M_n = f(x_0)$, for otherwise we have the following contradiction:

By the definition of $\lim M_n \neq f(x_0)$, $\exists \varepsilon_0 > 0$ and a subsequence (M_{n_k}) s.t. for all k,

$$|M_{n_k} - f(x_0)| \geq \varepsilon_0.$$

Then applying Bolzano-Weierstrass Theorem to (M_{n_k}), we have a convergent subsequence of (M_{n_k}), which is a convergent subsequence of (M_n) but its limit is not $f(x_0)$. This is a contradiction.

p.215 Q19. (3 marks)[‖]

(a) Since $f \in \mathcal{R}[-a,a]$, we have $f|_{[-a,0]} \in \mathcal{R}[-a,0]$, $f|_{[0,a]} \in \mathcal{R}[0,a]$, and $\int_{-a}^a f = \int_{-a}^0 f + \int_0^a f$. Therefore it suffices to show $\int_0^a f = \int_{-a}^0 f$. We do this by showing that $\forall\varepsilon > 0$,

$$\left| \int_0^a f - \int_{-a}^0 f \right| \leq \varepsilon.$$

Let $\varepsilon_0 > 0$. By using the tagged partition approach for Riemann integration, $\exists \delta_0 > 0$ s.t. for all tagged partition $\dot{P}$ of $[0,a]$ with

[‖]A student provides this solution.

$\left\|\dot{P}\right\| < \delta_0$, and all tagged partition $\dot{Q}$ of $[-a, 0]$ with $\left\|\dot{Q}\right\| < \delta_0$, we have

$$\left|S(f|_{[0,a]}, \dot{P}) - \int_0^a f\right| \le \frac{\varepsilon_0}{2} \text{ and}$$
$$\left|S(f|_{[-a,0]}, \dot{Q}) - \int_{-a}^0 f\right| \le \frac{\varepsilon_0}{2}.$$

Let $\dot{P}_0 = \{([x_{i-1}, x_i], s_i)\}_{i=1}^n$ be a tagged partition of $[0, a]$ with $\left\|\dot{P}_0\right\| < \delta_0$. Then $\dot{Q}_0 := \{([-x_i, -x_{i-1}], -s_i)\}_{i=n}^1$ is a tagged partition of $[-a, 0]$ with $\left\|\dot{Q}_0\right\| < \delta_0$. Note that $S(f|_{[0,a]}, \dot{P}_0) = S(f|_{[-a,0]}, \dot{Q}_0)$ because f is even. Hence

$$\begin{aligned}\left|\int_0^a f - \int_{-a}^0 f\right| &\le \left|\int_0^a f - S(f|_{[0,a]}, \dot{P}_0)\right| + \left|S(f|_{[-a,0]}, \dot{Q}_0) - \int_{-a}^0 f\right| \\ &\le \varepsilon_0,\end{aligned}$$

which was to be shown.

(b) Similar to (a), it suffices to show that $\forall \varepsilon > 0$,

$$\left|\int_0^a f + \int_{-a}^0 f\right| \le \varepsilon.$$

Let $\varepsilon_0 > 0$ and consider the same δ_0, $\dot{P}_0$, $\dot{Q}_0$ as in (a). Because f is odd, we have

$$\begin{aligned}S(f|_{[0,a]}, \dot{P}_0) &= \sum f(s_i)(x_i - x_{i-1}) \\ &= \sum -f(-s_i)(x_i - x_{i-1}) \\ &= -\sum f(-s_i)(-x_{i-1} - (-x_i)) \\ &= -S(f|_{[-a,0]}, \dot{Q}_0).\end{aligned}$$

Hence

$$\begin{aligned}\left|\int_0^a f + \int_{-a}^0 f\right| &\le \left|\int_0^a f - S(f|_{[0,a]}, \dot{P}_0)\right| + \left|-S(f|_{[-a,0]}, \dot{Q}_0) + \int_{-a}^0 f\right| \\ &\le \varepsilon_0.\end{aligned}$$

Done.