Tutorial 10 for MATH4220

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1. (a) Find the Fourier sine series of $\phi(x) = x$ on the interval $[0, l]$.
   (b) Find the Fourier cosine series of $\phi(x) = x$ on the interval $[0, l]$.
   (c) Find the full Fourier series of $\phi(x) = x$ on the interval $[-l, l]$.

   Solution: (a)The Fourier sine series of $\phi(x) = x$ is

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

where the coefficients are

$$\begin{aligned} B_n =& \frac{2}{l}\int_0^l x\sin(\frac{n\pi x}{l})dx \\ =& -\frac{2x}{n\pi}\cos(\frac{n\pi x}{l})\Big|_0^l + \frac{2}{n\pi}\int_0^l \cos(\frac{n\pi x}{l})dx \\ =& (-1)^{n+1}\frac{2l}{n\pi}, \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\frac{n\pi x}{l} = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots).$$

(b)The Fourier cosine series of $\phi(x) = x$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

where the coefficients are

$$A_0 = \frac{2}{l}\int_0^l x dx = l,$$

and

$$\begin{aligned} A_n =& \frac{2}{l}\int_0^l x\cos(\frac{n\pi x}{l})dx \\ =& \frac{2x}{n\pi}\sin(\frac{n\pi x}{l})\Big|_0^l - \frac{2}{n\pi}\int_0^l \sin(\frac{n\pi x}{l})dx \\ =& \frac{2l}{n^2\pi^2}\cos(\frac{n\pi x}{l})\Big|_0^l = \frac{2l}{n^2\pi^2}((-1)^n - 1), \quad n = 1, 2, \cdots \end{aligned}$$

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Hence

$$x = \frac{l}{2} - \sum_{n=1, n \text{ odd}}^{\infty} \frac{4l}{n^2\pi^2} \cos\frac{n\pi x}{l}.$$

(c) The full Fourier series of $\phi(x) = x$ is

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l})$$

where the coefficients are

$$A_n = \frac{1}{l}\int_{-l}^{l} x\cos(\frac{n\pi x}{l})dx = 0, \quad n = 0, 1, 2, \cdots$$

and

$$\begin{aligned} B_n =& \frac{1}{l}\int_{-l}^{l} x\sin(\frac{n\pi x}{l})dx \\ =& -\frac{x}{n\pi}\cos(\frac{n\pi x}{l})\Big|_{-l}^{l} + \frac{1}{n\pi}\int_{-l}^{l}\cos(\frac{n\pi x}{l})dx \\ =& (-1)^{n+1}\frac{2l}{n\pi}, \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2l}{n\pi}\sin\frac{n\pi x}{l} = \frac{2l}{\pi}(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots).$$

Remark: The full Fourier series and Fourier sine series of x are same, since x is odd.

2. **Gram-Schmidt orthogonalization procedure**

If $X_1, X_2, \cdots$ is an sequence (finite or infinite) of linearly independent vectors in any vector space with an inner product, it can be replaced by a sequence of linear combinations that are mutually othorgonal. The idea is that at each step one subtracts off the components parallel to the previuos vectors. More precisely,

$$\begin{aligned} Z_1 &= \frac{X_1}{\|X_1\|} \\ Z_2 &= \frac{Y_2}{\|Y_2\|}, Y_2 = X_2 - (X_2, Z_1)Z_1 \\ Z_3 &= \frac{Y_3}{\|Y_3\|}, Y_3 = X_3 - (X_3, Z_1)Z_1 - (X_3, Z_2)Z_2 \\ &\cdots \end{aligned}$$

(a) Show that all the vectors $Z_1, Z_2, Z_3, \cdots$ are orthoganal to each other.

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

Solution:

(a) Note that $(Z_1, Z_2) = \frac{1}{\|Y_2\|}(Z_1, X_2 - (X_2, Z_1)Z_1) = 0$.
Assume that any $Z_1, \cdots, Z_n$ are mutually orthogonal. Then for any $k \leq n$,

$$\begin{aligned}
(Z_{n+1}, Z_k) &= \frac{1}{\|Y_{n+1}\|}(X_{n+1} - (X_{n+1}, Z_1)Z_1 - \cdots - (X_{n+1}, Z_n)Z_n, Z_k) \\
&= \frac{1}{\|Y_{n+1}\|}((X_{n+1}, Z_k) - (X_{n+1}, Z_k)(Z_k, Z_k)) \\
&= 0.
\end{aligned}$$

(b) Here $X_1 = \cos x + \cos 2x$ and $X_2 = 3\cos x - 4\cos 2x$. Then

$$\begin{aligned}
Z_1 &= \frac{X_1}{\|X_1\|} = \frac{\cos x + \cos 2x}{\sqrt{\pi}} \\
Z_2 &= \frac{Y_2}{\|Y_2\|} = \frac{X_2 - (X_2, Z_1)Z_1}{\|Y_2\|} = \frac{\cos x - \cos 2x}{\sqrt{\pi}}.
\end{aligned}$$

3. Show that Robin boundary conditions are symmetric.

Solution: Suppose f and g are two functions satisfying Robin conditions

$$X'(0) - a_0 X(0) = 0, X'(l) + a_l X(l) = 0$$

then

$$\begin{aligned}
f'\bar{g} - f\bar{g}'\Big|_0^l &= f'(l)\overline{g(l)} - f(l)\overline{g'(l)} - f'(0)\overline{g(0)} + f(0)\overline{g'(0)} \\
&= -a_l f(l)\overline{g(l)} + a_l f(l)\overline{g(l)} - a_0 f(0)\overline{g(0)} + a_0 f(0)\overline{g(0)} = 0.
\end{aligned}$$