THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 3
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Dual space of $l^p(1 < p < \infty)$

Dual space. Let X be a normed space. Then the set of bounded linear functionals on X constitutes a normed space with norm defined by

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|,$$

which is called the dual space of X and denoted by X^*.

An isomorphism of a normed space X onto a normed space Y is a bijective linear operator $T : X \to Y$ which preserves the norm, that is, for all $x \in X$,

$$\|Tx\|_Y = \|x\|_X.$$

Our first theorem shows that the dual space of l^p is isomorphic with l^q. We express this by saying that the dual space of l^p is l^q.

Theorem. The dual space of l^p is the space l^q, where p, q are **Hölder conjugates**, i.e., $(l^p)^* \cong l^q, 1 < p < \infty$.

Proof: Step 1. $(l^p)^* \subset l^q$**.** Construct an injective operator

$$T : (l^p)^* \to l^q \quad s.t. \quad \|Tf\|_{l^q} \le \|f\|.$$

Let $f \in (l^p)^*$. For any $x \in l^p$, there exists a unique sequence of real numbers x_k such that $x = \sum_{k=1}^{\infty} x_k e_k$, where (e_k) is the Schauder basis of l^p. Then,

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k)$$

since f is continuous.

Denote $f(e_k)$ by b_k and we can define an injective linear operator T by $Tf = (b_k) = (f(e_k))$. It suffices to show that $(b_k) \in l^q$.

Indeed, $\forall n \in \mathbb{N}$, we can construct a sequence $x^n = (x_k^n)$ as

$$x_k^n = \begin{cases} \dfrac{|b_k|^q}{b_k}, & \text{if } b_k \neq 0 \text{ and } k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that $x^n \in l^p$ because it has finite nonzero terms and

$$f(x^n) = \sum_{k=1}^{\infty} x_k^n b_k = \sum_{k=1}^{n} |b_k|^q.$$

By the boundedness of f, we have

$$\sum_{k=1}^{n}|b_k|^q = |f(x^n)| \le \|f\|\|x^n\|_{l^p} = \|f\|\left(\sum_{k=1}^{\infty}|x_k^n|^p\right)^{\frac{1}{p}} = \|f\|\left(\sum_{k=1}^{n}|b_k|^{(q-1)p}\right)^{\frac{1}{p}} = \|f\|\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{p}}.$$

Therefore,

$$\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}} \le \|f\|.$$

Let $n \to \infty$ and we have $(b_k) \in l^q$ with

$$\|(b_k)\|_{l^q} \le \|f\|.$$

Step 2. $l^q \subset (l^p)^*$**.** To show that T is surjective and verify $\|Tf\|_{l^q} = \|f\|$.

For an arbitrary sequence $(b_k) \in l^q$, it can be checked that the mapping

$$f(x) := \sum_{k=1}^{\infty} x_k b_k, \quad \forall x = (x_k) \in l^p$$

is a bounded linear operator on l^p.

In fact,

$$|f(x)| = \left|\sum_{k=1}^{\infty} x_k b_k\right| \le \sum_{k=1}^{\infty}|x_k b_k| \le \|x\|_{l^p}\|(b_k)\|_{l^q},$$

which implies that $f \in (l^p)^*$ and $\|f\| \le \|(b_k)\|_{l^q}$.

Theorem. The dual space of l^1 is the space l^∞.

Theorem. The dual space of c and c_0 are both the space l^1.

The dual space of l^∞ is NOT l^1.

Proof. To be supplemented some time later.

Example. For $x = (x_n) \in l^2$, let $f(x) = \sum_{n=1}^{\infty} \frac{x_{2n}}{n}$, then $f \in (l^2)^*$.

Proof. f can be expressed as $f(x) = \sum_{k=1}^{\infty} b_k x_k$ with

$$b_k = f(e_k) = \begin{cases} \frac{1}{m}, & k = 2m, \\ 0, & k = 2m-1, \end{cases} \quad m \ge 1.$$

Then $f \in (l^2)^*$ and

$$\|f\| = \|(b_k)\|_2 = \left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}.$$

Theorem. Suppose $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space, $1 < p < \infty$. Then $\forall F \in (L^p(\Omega))^*$ can be written as

$$F(f) = \int_\Omega fg \, d\mu, \quad f \in L^p(\Omega)$$

where $g \in L^q(\Omega)$ is uniquely determined by F and $\|F\| = \|g\|_{L^q}$. In the sense of isomorphism, $(L^p(\Omega))^* = L^q(\Omega)$.

Example. Let $F(f) = \displaystyle\int_0^1 f(x^a)\, dx,\ f \in L^2[0,1], 0 < a < 2$. Then $F \in L^2[0,1]^*$.

Proof. By the substitution $x^a = t$, we have

$$F(f) = \int_0^1 f(t) \cdot \frac{1}{a} t^{\frac{1-a}{a}} \, dt.$$

Define $g(t) = \dfrac{t^{\frac{1-a}{a}}}{a}$. It can be checked that $g \in L^2[0,1]$. Therefore, $F \in L^2[0,1]^*$ and

$$\|F\| = \|g\|_{L^2} = \frac{1}{a} \left(\int_0^1 t^{\frac{2(1-a)}{a}} \, dt \right)^{\frac{1}{2}} = \frac{1}{\sqrt{a(2-a)}}.$$