THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 2
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Examples of linear operators

Operator norm. Given two normed spaces X and Y (over the same base field), a linear map $T : X \to Y$ is continuous if and only if it is bounded. One can show that the following definitions are all equivalent if $V \neq \{0\}$,

$$\begin{aligned} \|T\| &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \\ &= \sup\{\|Tv\| : x \in X \text{ with } \|x\| \leq 1\} \\ &= \sup\{\|Tv\| : x \in X \text{ with } \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \in X \text{ with } \|x\| \neq 0\right\}. \end{aligned}$$

It is important to keep in mind that the operator norm depends on our choice of norms for X and Y.

When studying a given operator $T : X \to Y$, we usually do three things,

- show that T is linear (usually trivial);
- estimate $\|Tx\|$, obtain some inequality $\|Tx\| \leq C\|x\|, \forall x \in X$ and conclude that $T \in B(X,Y)$ with $\|T\| \leq C$. The difficulty and techniques in this step depends on properties of the operator T;
- deduce that $\|T\| = C$. It suffices to show $\|T\| \geq C$, which is not always doable. Because when we enlarge $\|Tx\|$ in order to obtain the estimate $\|Tx\| \leq C\|x\|$, we may go beyond too much. That's to say, the estimate $\|Tx\| \leq C\|x\|$ is not optimal and the constant C is not exactly $\|T\|$.

  If we expect $\|T\|$ to be exactly C, we may try to find $x_0 \in X$ such that $\|x_0\| = a > 0, \|Tx_0\| \geq Ca$. Then we have

  $$Ca \leq \|Tx_0\| \leq \|T\|\|x_0\| = a\|T\| \Longrightarrow \|T\| \geq C.$$

  However, such x_0 may not exist. In this case, $\forall r \in (0, C)$, we find $x \in X$ with $\|x\| = 1, \|Tx\| \geq r$. We consequently conclude that $\|T\| \geq r$ and then let $r \to C$ to get $\|T\| \geq C$.

  For some complicated operators, it is difficult to find the exact value of $\|T\|$.

Examples

1. Let $X = (C[0,1], \|\cdot\|_\infty)$ and $(Tf)(x) = xf(x), (Sf)(x) = x\int_0^1 f(y)\,dy$. Find $\|T\|, \|S\|, \|TS\|, \|ST\|$.

Solution: It can be seen that T, S are both linear operators on $C[0,1]$.

- First, $\|Tf\|_\infty = \max_{x\in[0,1]} |xf(x)| \le \max_{x\in[0,1]} |f(x)| = \|f\|_\infty \Longrightarrow \|T\| \le 1$.

  Take $f \equiv 1$ and then $\|f\|_\infty = 1$ and $\|Tf\|_\infty = \max_{x\in[0,1]} |x| = 1 \le \|T\|\|f\|_\infty = \|T\|$. Therefore, $\|T\| = 1$.

- $\|Sf\|_\infty = \max_{x\in[0,1]} \left| x \int_0^1 f(y)\,dy \right| = \left| \int_0^1 f(y)\,dy \right| \le \|f\|_\infty \Longrightarrow \|S\| \le 1$.

  Take $f \equiv 1$ again as an example and we can also get $\|S\| \ge 1$.

- Since $(TSf)(x) = x^2 \int_0^1 f(y)\,dy$, we can compute $\|TS\| = 1$ in the same way.

- Notice that $(STf)(x) = x \int_0^1 yf(y)\,dy$,

  $\|STf\|_\infty = \left| \int_0^1 yf(y)\,dy \right| \le \int_0^1 y \max_{x\in[0,1]} f(y)\,dy = \frac{1}{2}\|f\|_\infty \Longrightarrow \|ST\| \le \frac{1}{2}$.

  Take $f \equiv 1$ and we can see $\|ST\| \ge \frac{1}{2}$.

Remark. We can see from this example that in general $\|ST\| \ne \|TS\|$ and $\|ST\| \le \|S\|\|T\|$.

2 (Generalization of example 4.6). Let $I = [a, b], Tf(x) = \int_a^x f(t)\,dt$. Then $T \in B(L^1(I), C(I))$ and $T \in B(L^1(I), L^1(I))$. Find $\|T\|$ for each case.

Solution: For the first case, we have

$$\|Tf\|_\infty = \max_{a\le x\le b} \left| \int_a^x f(t)\,dt \right| \le \max_{a\le x\le b} \int_a^x |f(t)|\,dt \le \int_a^b |f(t)|\,dt = \|f\|_1.$$

Hence $\|T\| \le 1$. Take $f \equiv 1$ and then $\|f\|_1 = b - a, Tf(x) = x - a$. Therefore,

$$b - a = \|Tf\|_\infty \le \|T\|\|f\|_1 = (b-a)\|T\| \Longrightarrow \|T\| \ge 1.$$

For the second case,

$$\begin{aligned} \|Tf\|_1 &= \int_a^b \left| \int_a^x f(t)\,dt \right| dx \le \int_a^b \int_a^x |f(t)|\,dt\,dx = \int_a^b \int_t^b |f(t)|\,dx\,dt \\ &= \int_a^b (b-t)|f(t)|\,dt \le \int_a^b (b-a)|f(t)|\,dt = (b-a)\|f\|_1 \Longrightarrow \|T\| \le b - a. \end{aligned}$$

Take $f_n(t) = \begin{cases} 1, & a \le t \le a + \frac{b-a}{n}, \\ 0, & a + \frac{b-a}{n} < t < b \end{cases}$ and then $\|f_n\|_1 = \frac{b-a}{n}, Tf_n(x) = \begin{cases} x - a, & a \le t \le a + \frac{b-a}{n}, \\ \frac{b-a}{n}, & a + \frac{b-a}{n} < t < b \end{cases}$.

Therefore,

$$\frac{(b-a)^2}{n} - \frac{(b-a)^2}{2n^2} = \|Tf_n\|_1 \le \|T\|\|f\|_1 = \frac{b-a}{n}\|T\| \Longrightarrow \|T\| \ge b - a - \frac{b-a}{2n}.$$

Let $n \to \infty$ and hence $\|T\| \ge b - a$.

Remark: We can see from this example that when regarded as operators on different normed spaces, the operator norm may differ.

3. For a infinite column vector $x = (x_1, x_2, \cdots)^T$, define $Ax = (2x_1, x_2 - x_1, x_3 - x_2, \cdots)$. Then $A \in B(l^1, l^1)$ and $A \in B(l^\infty, l^\infty)$. Find $\|T\|$ for each case.

Solution: For $T \in B(l^1, l^1)$,

$$\begin{aligned}\|Ax\|_1 &= |2x_1| + |x_2 - x_1| + |x_3 - x_2| + \cdots \leq 2|x_1| + (|x_1| + |x_2|) + (|x_2| + |x_3|) \cdots \\ &= 3|x_1| + 2|x_2| + 2|x_2| + \cdots \leq 3\|x\|_1 \Longrightarrow \|A\| \leq 3.\end{aligned}$$

Consider $x = e_1 = (1, 0, 0, \cdots)$ and we can conclude $\|A\| \geq 3$.

For $A \in B(l^\infty, l^\infty)$,

$$\begin{aligned}\|Ax\|_\infty &= \sup\{|2x_1|, |x_2 - x_1|, \cdots\} \leq \sup\{|2x_1|, |x_2| + |x_1|, \cdots\} \\ &\leq 2\sup\{|x_1|, |x_2|, \cdots\} = 2\|x\|_\infty \Longrightarrow \|A\| \leq 2.\end{aligned}$$

Take $x = (1, 0, 0, \cdots)$ and we can see that $\|A\| \geq 2$.

4. Examples 4.4-4.5 in the text.

General cases. 1. Suppose $A = (a_{ij})$ is an infinite matrix and $x = (x_1, x_2, \cdots)^T$. We formally define

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \\ \cdots & \cdots & \ddots & \vdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$$

as normal matrix product, which means $Ax = (y_1, y_2, \cdots)^T$ where

$$y_i = \sum_j a_{ij} x_j, \quad i = 1, 2, \cdots .$$

Then we have

- if $C := \sup_j \sum_i |a_{ij}| < \infty$, then $A \in B(l^1, l^1)$ and $\|A\|_1 = C$.
- if $C := \sup_i \sum_j |a_{ij}| < \infty$, then $A \in B(l^\infty, l^\infty)$ and $\|A\|_\infty = C$.
- if $C := \sum_{i,j} |a_{ij}|^2 < \infty$, then $A \in B(l^2, l^2)$ and $\|A\| \leq C^{\frac{1}{2}}$.

  Remark: When A is finite, we know $\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$.

2 (Generalization of example 4.3). Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, the operator T is defined as

$$(Tf)(x) = \int_\Omega k(x,y) f(y)\, d\mu(y), \quad x \in \Omega$$

where $k(x,y)$ is a $\mu \times \mu$-measurable function on Ω, then we have

- if $C := \left\| \int_\Omega |k(x,y)|\, d\mu(x) \right\|_\infty < \infty$, then $T \in B(L^1(\Omega), L^1(\Omega))$ and $\|T\| = C$.
- if $C := \left\| \int_\Omega |k(x,y)|\, d\mu(y) \right\|_\infty < \infty$, then $T \in B(L^\infty(\Omega), L^\infty(\Omega))$ and $\|T\| = C$.
- if $C := \int_{\Omega \times \Omega} |k(x,y)|^2\, d\mu(x)\, d\mu(y) < \infty$, then $T \in B(L^2(\Omega), L^2(\Omega))$ and $\|T\| \leq C^{\frac{1}{2}}$.
- if $\Omega \subset \mathbb{R}^n$ is bounded and closed, and $k(x,y) \in C(\Omega)$, then $T \in B(C(\Omega), C(\Omega))$ and

$$\|T\| = \max_{x \in \Omega} \int_\Omega |k(x,y)|\, dy.$$