MATH2060B TUTORIAL 5

For Riemann Integration Theory, we will follow closely to the notes uploaded in the course webpage instead of the textbook.

Construction of Riemann Integral:

* We always consider bounded functions $f, g, h, \ldots$ etc defined on a closed bounded interval $[a, b]$, and let m and M be an lower and upper bound of f respectively. i.e., $m \leq f(x) \leq M$ for any $x \in [a, b]$.
* A partition P of the interval $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \ldots\ldots < x_n = b.$$

* For any partition P of $[a, b]$, denote
    - $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \ldots, n$
    - $\|P\| = \max \Delta x_i$
* For any partition P of $[a, b]$ and function f defined on $[a, b]$, denote
    - $m_i(f, P) = \inf\{ f(x) : x \in [x_{i-1}, x_i] \}$.
    - $M_i(f, P) = \sup\{ f(x) : x \in [x_{i-1}, x_i] \}$.

    Always exist because f is bounded!

    - $\omega_i(f, P) = M_i(f, P) - m_i(f, P) = \sup\{ |f(x) - f(y)| : x, y \in [x_{i-1}, x_i] \}$.

    Why?

    - (Lower sum) $L(f, P) = \sum m_i(f, P) \Delta x_i$
    - (Upper sum) $U(f, P) = \sum M_i(f, P) \Delta x_i$

    We always have $m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$ for any partition P.

* For any function f defined on $[a, b]$, denote
    - (Lower integral) $\underline{\int_a^b} f = \sup\{ L(f, P) : P \text{ is a partition of } [a, b] \}$
    - (Upper integral) $\overline{\int_a^b} f = \inf\{ U(f, P) : P \text{ is a partition of } [a, b] \}$

    Always exists by this observation!

* If f has equal upper and lower integral, we say that f is Riemann integrable. We write $f \in R[a, b]$ in this case and

$$\underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f \qquad \text{(integral of } f\text{)}$$

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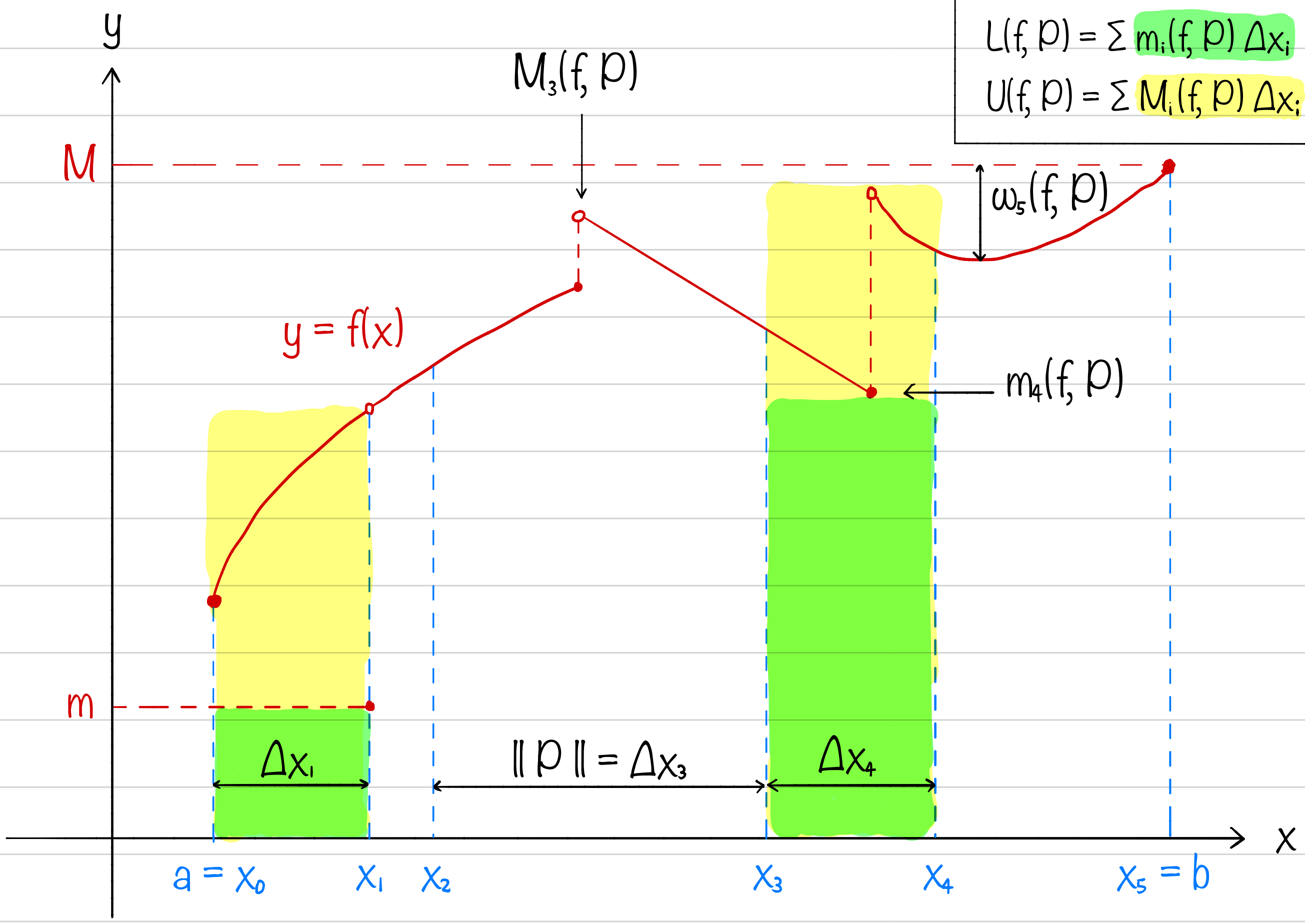
Example 1: Visualize the notations in a picture.

$P = \{a, x_1, x_2, x_3, x_4, b\}$

$L(f, P) = \sum m_i(f, P)\,\Delta x_i$

$U(f, P) = \sum M_i(f, P)\,\Delta x_i$



Example 2: Show that the function $f(x) = x$ is Riemann integrable on $[0, 1]$.

Solution: We need to show that f has the same lower and upper integrals.

For each $n \in \mathbb{N}$, consider the partition P_n of $[0, 1]$ defined by

$$P_n = \{0, 1/n, 2/n, \dots, 1\}.$$

On each subinterval $[(i-1)/n, i/n]$, where $i = 1, 2, \dots, n$, we have

$$\Delta x_i = 1/n, \quad m_i(f, P_n) = (i-1)/n \quad \text{and} \quad M_i(f, P_n) = i/n$$

Compute the corresponding upper and lower sum:

$$L(f, P_n) = \sum m_i \Delta x_i = \frac{1}{n^2}\sum (i-1) = \frac{1}{n^2}\,\frac{n(n-1)}{2} = \frac{1}{2}(1 - 1/n)$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

$$U(f, P_n) = \sum M_i \Delta x_i = \frac{1}{n^2}\sum i = \frac{1}{n^2}\,\frac{n(n+1)}{2} = \frac{1}{2}(1 + 1/n)$$

It follows that $\displaystyle \frac{1}{2}(1 - 1/n) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \frac{1}{2}(1 + 1/n)$, for all $n \in \mathbb{N}$

Letting $n \to \infty$, we conclude that f is Riemann integrable with integral $1/2$. #

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Example 2: Show that the Dirichlet's function is not Riemann integrable on $[0, 1]$.

Solution: Recall that the Dirichlet's function is defined by

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

We need to show that it has unequal upper and lower integrals.

Fix any partition P of $[0, 1]$. On each subinterval $[x_{i-1}, x_i]$, we have

$$m_i(g, P) = 0 \quad \text{and} \quad M_i(g, P) = 1$$

(We don't know about Δx_i!)

Then $L(g, P) = \sum m_i \Delta x_i = 0$, and $U(g, P) = \sum M_i \Delta x_i = \sum \Delta x_i = 1$.

Since the partition P is arbitrary, it follows that

$$\underline{\int_a^b} f = 0 \quad \text{and} \quad \overline{\int_a^b} f = 1.$$

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Useful Propositions:

- Let f be a function defined on $[a, b]$ and P, Q be partitions of $[a, b]$.
  - If $P \subseteq Q$, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.
  - $L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, Q)$

- Let f be a bounded function defined on $[a, b]$. Then $f \in R[a, b]$ if and only if for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) = \sum \omega_i(f, P)\Delta x_i < \varepsilon.$$

- $C[a, b] \subseteq R[a, b]$.

Remark: Let me say more about the proof of Lemma 1.2 (i) in the notes. It claims that it suffices to show the case that $Q = P \cup \{c\}$. i.e., Q contains exactly one more point than P. Here is why: Suppose in general that Q contains k more points than P. i.e., $Q = P \cup \{c_1, c_2, \ldots, c_k\}$. If we write

$$Q_1 = P \cup \{c_1\}, \; Q_2 = Q_1 \cup \{c_2\}, \; \ldots, \; Q = Q_k = Q_{k-1} \cup \{c_k\}.$$

Then by applying the special case k times, we have

$$L(f, P) \leq L(f, Q_1) \leq L(f, Q_2) \leq \cdots \leq L(f, Q_k) = L(f, Q).$$

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Exercises:

1. Suppose that f is a continuous and non-negative function defined on $[a, b]$. If the integral of f is 0, show that f is constantly zero.

Solution: We show the assertion by contradiction.

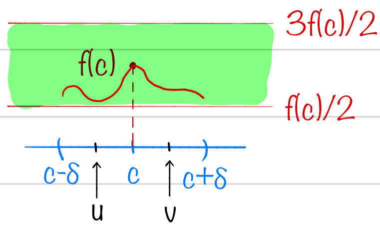
Suppose on a contrary that $f(c) > 0$ for some $c \in [a, b]$. Since f is continuous at c, there exists $\delta > 0$ such that whenever $|x - c| < \delta$,

$$|f(x) - f(c)| < f(c)/2. \quad \text{(This act as } \varepsilon)$$

i.e., $$0 < f(c)/2 < f(x) < 3f(c)/2 \quad \text{on } (c-\delta, c+\delta)$$

Now consider the partition P of $[a, b]$ defined by $P = \{a, u, v, b\}$, where

$$a < c-\delta < u < v < c+\delta < b$$



Then we can compute the lower sum:

$$\begin{aligned} L(f, P) &= m_1(f, P)\Delta x_1 + m_2(f, P)\Delta x_2 + m_3(f, P)\Delta x_3 \\ &\geq 0 + m_2(f, P)\Delta x_2 + 0 \\ &> 0 \end{aligned}$$

(≥ 0 is not enough in this proof!)

It is a contradiction because

$$0 = \int_a^b f = \underline{\int_a^b} f \geq L(f, P) > 0$$

given $\quad f \in R[a,b] \quad$ sup

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Remark: Can the continuity of f be dropped?

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