MATH2060B TUTORIAL 3

Announcement:

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* Check the course webpage frequently!
* Tutorial will be the same for Monday and Wednesday session

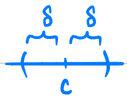
Send HW in ONE PDF file to him!

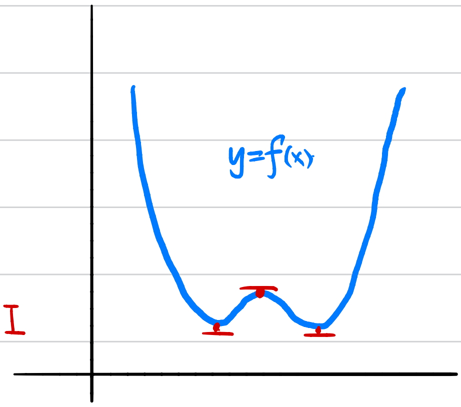
Mean Value Theorem:

Definition: Let $f: I \to \mathbb{R}$ be a function.

* f is said to have an absolute/global maximum at $c \in I$
  if $f(c) \geq f(x)$ $\forall x \in I$
* f is said to have an relative/local maximum at $c \in I$
  if $\exists$ neighbourhood $V = V_\delta(c)$ s.t. $f(c) \geq f(x)$ $\forall x \in V \cap I$

$(c-\delta, c+\delta)$





Reading Exercise:

6.2.1 Interior Extremum Theorem *Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.*

6.2.2 Corollary *Let $f: I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.*

6.2.3 Rolle's Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

6.2.4 Mean Value Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a).$$

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Applications:

(I) Determine whether a differentiable function is increasing or decreasing

Definition: A function $f: I \to \mathbb{R}$ is said to be increasing on I if $\forall x, y \in I$ with $x < y$, $f(x) \leq f(y)$.

6.2.7 Theorem *Let $f: I \to \mathbb{R}$ be differentiable on the interval I. Then:*

(a) *f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.*
(b) *f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.*

(II) First Derivative Test for relative extrema

6.2.8 First Derivative Test for Extrema *Let f be continuous on the interval $I := [a, b]$ and let c be an interior point of I. Assume that f is differentiable on (a, c) and (c, b). Then:*

(a) *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \geq 0$ for $c - \delta < x < c$ and $f'(x) \leq 0$ for $c < x < c + \delta$, then f has a relative maximum at c.*
(b) *If there is a neighborhood $(c - \delta, c + \delta) \subseteq I$ such that $f'(x) \leq 0$ for $c - \delta < x < c$ and $f'(x) \geq 0$ for $c < x < c + \delta$, then f has a relative minimum at c.*

(III) Deduce useful inequalities

Example 1: Show that $-x \leq \sin x \leq x \quad \forall x \geq 0$.

Solution: Let $x \geq 0$.

Case 1: If $x = 0$. Then $-x = \sin x = x = 0$.

Case 2: If $x > 0$. Then consider the sine function.

Note that it is continuous on $[0, x]$ and differentiable on $(0, x)$.

Hence by MVT, there exist $c \in (0, x)$ such that

$$\sin x - \sin 0 = (\cos c)(x - 0)$$

$$\sin x = x \cos c \quad \text{———} \quad (*)$$

Note that since $-1 \leq \cos c \leq 1$ and $x \geq 0$, we have

$$-x \leq x \cos c \leq x.$$

i.e., by $(*)$, $\quad -x \leq \sin x \leq x.$ #

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Example 2: Show that $\dfrac{x-1}{x} < \ln x < x - 1 \quad \forall\, x > 1$.

Solution: Let $x > 1$. Consider the function ln.

Note that it is continuous on $[1, x]$ and differentiable on $(1, x)$.

Hence by MVT, there exist $c \in (1, x)$ such that

$$\ln x - \ln 1 = (1/c)(x - 1)$$

$$\ln x = (x-1)/c \qquad \text{———} \quad (*)$$

Note that since $0 < 1 < c < x$ and $x - 1 > 0$, we have

$$(x-1)/x < (x-1)/c < (x-1)/1.$$

i.e., by $(*)$, $\qquad (x-1)/x < \ln x < x - 1.$ $\qquad$ #

(IV) Approximations

Example: Without Using a calculator, correct $\sqrt{105}$ to 1 decimal places.

Solution: Consider the square root function.

Note that it is continuous on $[100, 105]$ and differentiable on $(100, 105)$.

Hence by MVT, there exist $c \in (100, 105)$ such that

$$\sqrt{105} - \sqrt{100} = (1/2\sqrt{c})(105 - 100)$$

$$\sqrt{105} = 10 + 5/2\sqrt{c} \qquad \text{———} \quad (*)$$

Note that since $100 < c < 105 \boxed{< 121}$, we have $10 < \sqrt{c} \boxed{< 11}$.

Hence together with $(*)$,

$$10 + 5/2(11) < \sqrt{105} < 10 + 5/2(10) \qquad \text{———} \quad (\#)$$

Note that $10 + 5/2(11) = 225/22 = 10.227\ldots > 10.22$

$$10 + 5/2(10) = 45/4 = 10.25$$

Thus, $10.22 < \sqrt{105} < 10.25$, therefore $\sqrt{105} = 10.2$ (1 d.p.)

Remark: Using the upper bound 10.25 of $\sqrt{105}$, we can replace the 11 in $(\#)$

because $\sqrt{c} < \sqrt{105} < 10.25$. We then get a better lower bound

$$10 + 5/2(10.25) = 420/41 = 10.243\ldots > 10.24.$$

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Exercises:

1. **Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.**

Solution: We need to show by definition that

$$\boxed{\forall \varepsilon > 0,\ \exists a \in \mathbb{R} \text{ such that whenever } x > a,\ |g(x) - 0| < \varepsilon.}$$

Let $\varepsilon > 0$. Since $f'(x) \to 0$ as $x \to +\infty$, there exists $a \in \mathbb{R}$ such that

$$\text{whenever } x > a, \quad |f'(x) - 0| < \varepsilon. \quad \text{——— } (*)$$

Now fix any $x > a$. Note that f is continuous on $[x, x+1]$ and differentiable on $(x, x+1)$. By MVT, there exists $c \in (x, x+1)$ such that

$$g(x) \underset{\text{definition}}{=} f(x+1) - f(x) \underset{\text{MVT}}{=} f'(c)((x+1) - x) = f'(c) \quad \text{——— } (\#)$$

Also note that $c > x > a$. Thus, $(*)$ holds for c. It follows that

$$|g(x) - 0| \underset{(\#)}{=} |f'(c) - 0| \underset{(*)}{<} \varepsilon. \qquad \#$$

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2. **Suppose g is a real function on R^1, with bounded derivative (say $|g'| \le M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)**

Solution: We need to show that

$$\boxed{\exists\, \eta > 0 \text{ such that whenever } 0 < \varepsilon < \eta,\ f \text{ is one-to-one.}}$$

Take $\eta = 1/M$. We now show that f is one-to-one whenever $0 < \varepsilon < \eta$.

Suppose $f(x) = f(y)$. Assume it were true that $x < y$.

Note that g is continuous on $[x, y]$ and differentiable on (x, y).

By MVT, there exists $c \in (x, y)$ such that

$$g(y) - g(x) = g'(c)(y - x) \qquad (*)$$

Now from $f(x) = f(y)$, we can deduce that

$$x + \varepsilon g(x) = y + \varepsilon g(y)$$

$$x - y = \varepsilon\,(g(y) - g(x))$$

$$x - y \overset{(*)}{=} \varepsilon\, g'(c)(y - x)$$

Since $x \neq y$ and $\varepsilon \neq 0$, we have

$$g'(c) = -1/\varepsilon$$

Hence $M \geq |g'(c)| = 1/\varepsilon$. i.e., $\varepsilon \geq 1/M = \eta$. Contradiction!

It follows that we must have $x \geq y$. Similarly, we also have $x \leq y$.

Therefore $x = y$. #

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