Question 1. Let

$$f(x) = \sum_{k=1}^{\infty} \left(x + \frac{1}{k}\right)^k.$$

(a) Find the convergence domain D of f. i.e., $D = \{x \in \mathbb{R} : f(x) \text{ is convergent}\}$.

(b) Does the function f converge uniformly on its domain?

(c) Is f a $\mathcal{C}^1$-function on its domain?

(d) Show that the domain of f is equal to the domain of its derivatives f'.

Solution. .

(a) The domain of convergence is given by $D = (-1, 1)$. For each $k \in \mathbb{N}$, put

$$u_k(x) = \left(x + \frac{1}{k}\right)^k \quad \text{and hence} \quad f(x) = \sum_{k=1}^{\infty} u_k(x).$$

Notice that for each $x \in \mathbb{R}$,

$$\lim_{k\to\infty} |u_k(x)|^{1/k} = \lim_{k\to\infty} \left|\left(x + \frac{1}{k}\right)^k\right|^{1/k} = \lim_{k\to\infty} \left|x + \frac{1}{k}\right| = |x|.$$

Hence by the **Root Test**, $f(x)$ is (absolutely) convergent if $|x| < 1$ and is divergent if $|x| > 1$. It remains to consider the cases $x = \pm 1$.

If $x = 1$, note that each term of the series is bounded below by 1:

$$u_k(1) = \left(1 + \frac{1}{k}\right)^k \geq 1 > 0, \quad \forall k \in \mathbb{N}.$$

Hence $f(1)$ is divergent by the **Comparison Test**.

If $x = -1$, note that

$$u_k(-1) = \left(-1 + \frac{1}{k}\right)^k = (-1)^k \left(1 - \frac{1}{k}\right)^k, \quad \forall k \in \mathbb{N}.$$

By considering the odd subsequence and even subsequence of $(u_k(-1))$, we have

$$\lim_{k\to\infty} u_{2k}(-1) = \frac{1}{e} \quad \text{and} \quad \lim_{k\to\infty} u_{2k+1}(-1) = -\frac{1}{e}.$$

Hence $u_k(-1)$ is divergent. In particular, it does not converge to 0. It follows that $f(-1)$ is divergent by the n**-th Term Test**.

Remark. It is not enough to show that $f(x)$ is convergent for $x \in (-1, 1)$. This only implies that $(-1, 1) \subseteq D$. We should also show that $f(x)$ is divergent for $x \notin (-1, 1)$ to argue for $D = (-1, 1)$.

(b) No, f does not converge uniformly on its domain. By **Cauchy Criterion for Series**, it suffices to show that there exists an $\varepsilon > 0$ such that for all $K \in \mathbb{N}$, there exist $k \geq K$, $p \in \mathbb{N}$ and $x \in (-1, 1)$ such that

$$|u_{k+1}(x) + u_{k+2}(x) + \cdots + u_{k+p}(x)| \geq \varepsilon.$$

Fix any $K \in \mathbb{N}$, we can take

$$k = K, \quad p = 1, \quad \text{and} \quad x = \frac{K}{K+1}.$$

Then $k \geq K$, $p \in \mathbb{N}$ and $x \in (-1, 1)$. Also,

$$|u_{k+1}(x)| = \left| \left(x + \frac{1}{k+1} \right)^{k+1} \right| = \left| \left(\frac{K}{K+1} + \frac{1}{K+1} \right)^{K+1} \right| = 1^{K+1} \geq 1.$$

(c) Yes, f is a $\mathcal{C}^1$-function. Consider the function g defined by

$$g(x) = \sum_{k=1}^{\infty} u_k'(x) = \sum_{k=1}^{\infty} k \left(x + \frac{1}{k} \right)^{k-1}.$$

It suffices to show that g is uniformly convergent on $[-\eta, \eta]$ whenever $0 < \eta < 1$. Let $\eta < r < 1$. Then whenever $x \in [-\eta, \eta]$ and k sufficiently large, we have

$$|u_k'(x)| \leq k \left(|x| + \frac{1}{k} \right)^{k-1} \leq k \left(\eta + \frac{1}{k} \right)^{k-1} \leq k \cdot r^{k-1}.$$

Notice that $\sum kr^{k-1}$ is convergent by the **Ratio Test**. It follows by the **Weierstrass M-Test** that $g(x)$ is uniformly convergent on $[-\eta, \eta]$.

For any $x \in (-1, 1)$, pick η such that $|x| < \eta < 1$. Since $g(x)$ is uniformly convergent on $[-\eta, \eta]$ and $x \in [-\eta, \eta]$, we have

$$f'(x) = g(x) = \sum_{k=1}^{\infty} u_k'(x) = \sum_{k=1}^{\infty} k \left(x + \frac{1}{k} \right)^{k-1}, \quad \forall x \in (-1, 1).$$

Note that each u_k' is continuous at x, so $f' = g$ is also continuous at x. Since $x \in (-1, 1)$ is arbitrary, it follows that f is a $\mathcal{C}^1$-function.

(d) As shown in (c), the domain of f and f' are both $(-1, 1)$.

Question 2. .

(a) Show that the following statements are equivalent without using Cauchy criterion:

(i) Every Cauchy sequence in $\mathbb{R}$ is convergent.

(ii) Every absolutely convergent series in $\mathbb{R}$ is convergent.

(a) Give an example to show that statement (ii) above does not hold if $\mathbb{R}$ is replaced by the field of rational numbers $\mathbb{Q}$.

Solution. .

(a) (i $\Rightarrow$ ii) Let $\sum x_n$ be an absolutely convergent series in $\mathbb{R}$. For each $n \in \mathbb{N}$, denote

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + \cdots + x_n$$

be the partial sums. Then for $n, p \in \mathbb{N}$,

$$|s_{n+p} - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| \le \sum_{k=n+1}^{n+p} |x_k|.$$

Since $\sum |x_n|$ is convergent, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{n+p} |x_k| < \varepsilon, \quad \forall n \ge N, \forall p \in \mathbb{N}.$$

It follows that (s_n) is a Cauchy sequence in $\mathbb{R}$, and hence converges to some $x \in \mathbb{R}$ by assumption (i). Therefore the absolutely convergent series $\sum x_n$ is convergent.

(ii $\Rightarrow$ i) Let (x_n) be a Cauchy sequence in $\mathbb{R}$. It suffices to show that (x_n) admits a convergent subsequence, which is constructed in the following way:

For $k = 1$, since (x_n) is a Cauchy sequence, there exists $n_1 \in \mathbb{N}$ such that

$$|x_m - x_{n_1}| < \frac{1}{2}, \quad \forall m \ge n_1.$$

For $k = 2$, since (x_n) is a Cauchy sequence, there exists $n_2 > n_1$ such that

$$|x_m - x_{n_2}| < \frac{1}{2^2}, \quad \forall m \ge n_2.$$

Repeat the constructions for $k = 3, 4, \ldots$, we defined a subsequence (x_{n_k}) such that

$$|x_{n_{k+1}} - x_{n_k}| < \frac{1}{2^k} \quad \forall k \in \mathbb{N}.$$

Let $y_k = x_{n_{k+1}} - x_{n_k}$. Note that for each $K \in \mathbb{N}$,

$$\sum_{k=1}^{K} |y_k| = \sum_{k=1}^{K} |x_{n_{k+1}} - x_{n_k}| < \sum_{k=1}^{K} \frac{1}{2^k} \le \sum_{n=1}^{\infty} \frac{1}{2^k} = 1.$$

i.e., $\sum y_k$ is absolutely convergent, and hence converges to some $y \in \mathbb{R}$ by assumption (ii). Now, consider the partial sums of $\sum y_k$, we have

$$\sum_{k=1}^{K} y_k = (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots + (x_{n_{K+1}} - x_{n_K}) = x_{n_{K+1}} - x_{n_1}.$$

Taking limit as $K \to \infty$, we see that (x_{n_K}) is convergent:

$$\lim_{K\to\infty} x_{n_K} = \lim_{K\to\infty} x_{n_{K+1}} = x_{n_1} + \lim_{K\to\infty} \sum_{k=1}^{K} y_k = x_{n_1} + y$$

(b) **(Method I)** Fix an irrational number, say $\sqrt{2}$, and consider its decimal representation:

$$\sqrt{2} = 1.414121356....$$

For each $n \in \mathbb{N}$, define the sequence (u_n) in $\mathbb{Q}$ by truncating the first n decimal places:

$$u_1 = 1.4, \quad u_2 = 1.41, \quad u_3 = 1.414, \quad ...$$

If we define $x_1 = u_1$ and $x_{n+1} = u_{n+1} - u_n$ for $n \in \mathbb{N}$, then $\sum x_n$ is a series in $\mathbb{Q}$ that satisfies

$$\sum_{n=1}^{\infty} |x_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} x_n = \sqrt{2}.$$

(Method II) Fix an irrational number $\alpha \in (-1, 1)$. We construct the sequence of signs (ε_n) inductively in the following way:

For $n = 1$, let $\alpha_1 = \alpha$. Notice that α_1 is an irrational number because α is irrational. Note that we also have $0 < |a_1| < 1$. Define ε_1 to be the same sign as α_1. i.e.,

$$\varepsilon_1 = \begin{cases} 1, & \text{if } \alpha_1 > 0, \\ -1, & \text{if } \alpha_1 < 0. \end{cases}$$

For $n = 2$, let $\alpha_2 = \alpha_1 - \varepsilon_1/2$. Notice that α_2 is an irrational number because α_1 is irrational. Note that we also have $0 < |a_2| < \frac{1}{2}$. Define ε_2 to be the same sign as α_2.

Repeat the constructions for $n = 3, 4, ...$, we defined a sequence (α_n) and a sequence of signs (ε_n) such that

$$\alpha_{n+1} = \alpha_n - \frac{\varepsilon_n}{2^n} = \alpha - \sum_{k=1}^{n} \frac{\varepsilon_k}{2^k} \quad \text{and} \quad 0 < |\alpha_n| < \frac{1}{2^{n-1}}, \quad \forall n \in \mathbb{N}.$$

It follows that the series $\sum \varepsilon_n 2^{-n}$ is an absolutely convergent series in $\mathbb{Q}$:

$$\sum_{n=1}^{\infty} \left|\frac{\varepsilon_n}{2^n}\right| = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$$

However, it is not a convergent series in $\mathbb{Q}$ because

$$\left|\alpha - \sum_{k=1}^{n} \frac{\varepsilon_k}{2^k}\right| = |\alpha_{n+1}| < \frac{1}{2^n} \to 0, \quad \text{as } n \to \infty.$$

i.e., $\sum \varepsilon_n 2^{-n}$ converges to $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.