THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 9: Section 9.1: 7, 10, 13

7. (a) If $\sum a_n$ is absolutely convergent and (b_n) is a bounded sequence, show that $\sum a_n b_n$ is absolutely convergent.
   (b) Give an example to show that if the convergence of $\sum a_n$ is conditional and (b_n) is a bounded sequence, then $\sum a_n b_n$ may diverge.

(3 marks)

Solution.

(a) Since $\sum a_n$ is absolutely convergent, we have $\displaystyle\sum_{n=1}^{\infty} |a_n| < \infty$. Let $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} |a_n b_n| \leq \sum_{n=1}^{\infty} M|a_n| = M\left(\sum_{n=1}^{\infty} |a_n|\right) < \infty.$$

Therefore, $\sum a_n b_n$ is absolutely convergent.

(b) Let $a_n = (-1)^n/n$ and $b_n = (-1)^n$. By **9.3.2 Alternating Series Test**, we see that $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. On the other hand,

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

The alternating series test states that if (a_n) is a sequence of positive numbers, which is monotone decreasing with $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$ exists. To see that, if we put $S_n = (-1)a_1 + a_2 + \cdots + (-1)^n a_n$ the partial sum, then $(S_{2n-1})_{n\in\mathbb{N}}$ is an increasing sequence bounded above by 0, (why?) and (S_{2n}) is a decreasing sequence bounded below by $-a_1$ (why?). Assume that they converge to the limits l_1, l_2 respectively. Then,

$$|l_2 - l_1| = \lim_{n\to\infty} |S_{2n} - S_{2n-1}| = \lim_{n\to\infty} |a_{2n}| = 0$$

Some routine argument shows that $\lim_{n\to\infty} S_n = l_1$, i.e. $\sum_{n=1}^{\infty}(-1)^n a_n$ exists. One may show that $\sum_{n=1}^{\infty} 1/n = \infty$ by integral test. (9.2.6)

10. Give an example of a divergent series $\sum a_n$ with (a_n) decreasing and such that $\lim(na_n) = 0$.

(3 marks)

Solution. Let $a_n = \frac{1}{(n+1)\log(n+1)}$ for $n \in \mathbb{N}$. This is to avoid the case $\log 1 = 0$. It is easy to see that (a_n) is a decreasing sequence with $\lim_{n\to\infty} na_n = 0$. To show that the infinite sum is divergent, we may appy the integral test, by termwise comparison, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} &\geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{(x+1)\log(x+1)}\, dx \\ &= \int_1^{\infty} \frac{1}{(x+1)\log(x+1)}\, dx \\ &= \int_1^{\infty} \frac{d(\log(x+1))}{\log(x+1)} \\ &= \log(\log(x+1))|_{x=1}^{\infty} = \infty\end{aligned}$$

13. (a) Does the series $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \right)$ converge?

(b) Does the series $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{n} \right)$ converge?

(2 marks each)

Solution.

(a) No. Notice that

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} &= \frac{1}{\sqrt{n}(\sqrt{n+1}+\sqrt{n})} \\ &\geq \frac{1}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n+1})} \\ &= \frac{1}{2(n+1)}\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} \right) \geq \sum_{n=1}^{\infty} \frac{1}{2(n+1)} = \infty.$$

(b) Yes. Notice that

$$\begin{aligned}\frac{\sqrt{n+1}-\sqrt{n}}{n} &= \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \\ &\leq \frac{1}{n(\sqrt{n}+\sqrt{n})} \\ &= \frac{1}{2n\sqrt{n}}\end{aligned}$$

We may conclude that

$$\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{n} \right) \leq \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}} < \infty.$$

using integral test.