THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 6: Section 7.2: 16, 17

16. If f is continuous on $[a,b]$, $a<b$, show that there exists $c \in [a,b]$ such that we have $\int_a^b f = f(c)(b-a)$. This result is sometimes called the *Mean Value Theorem for Integrals.* (3 marks)

Solution.

Let $F(x) = \int_a^x f$ for $x \in [a,b]$. The function F is continuous on $[a,b]$ and by Fundamental theorem of Calculus (Second form), F is differentiable on (a,b) with $F'(x) = f(x)$. By Mean Value Theorem, there is some $c \in (a,b)$ such that

$$F(b) - F(a) = F'(c)(b-a)$$

This completes the proof.

17. If f and g are continuous on $[a,b]$ and $g(x) > 0$ for all $x \in [a,b]$, show that there exists $c \in [a,b]$ such that $\int_a^b fg = f(c)\int_a^b g$. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.) (7 marks)

Solution. Since g is continuous on $[a,b]$, there is some $x_0 \in [a,b]$ such that $g(x) \geq g(x_0)$ for every $x \in [a,b]$. By assumption, $g(x_0) > 0$, hence $\int_a^b g \geq g(x_0)(b-a) > 0$.

Let $M := \sup\{f(x) : x \in [a,b]\}$ and $m := \inf\{f(x) : x \in [a,b]\}$. For each $x \in [a,b]$, we have

$$mg(x) \leq f(x)g(x) \leq Mg(x),$$

because $g(x) > 0$. Integrating from a to b gives

$$m\int_a^b g \leq \int_a^b fg \leq M\int_a^b g.$$

Divided by $\int_a^b g\,(>0)$, we see that $\frac{\int_a^b fg}{\int_a^b g} \in [m,M]$. Intermediate Value Theorem tells us that $\frac{\int_a^b fg}{\int_a^b g} = f(c)$ for some $c \in [a,b]$. This shows the first part. (4 marks)

To see that the conclusion fails if we do not have $g(x) > 0$, let $g(x) = f(x) = x$ on $[-1,1]$. Then, $\int_{-1}^1 fg = \int_{-1}^1 x^2 = \frac{2}{3}$, but $\int_{-1}^1 g = 0$. Therefore, we cannot have $\int_{-1}^1 fg = f(c)\int_{-1}^1 g$ for any $c \in [-1,1]$. (3 marks)