THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 11: Section 9.4: 16, 17, 19

16. Show by integrating the series for $1/(1+x)$ that if $|x| < 1$, then

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

(3 marks)

Solution. For each $a \in (0,1)$, notice that the series $\sum_{n=0}^{\infty} (-1)^n x^n$ converges uniformly to $1/(1+x)$ on $[-a,a]$. Therefore, for any $x \in [-a,a]$, we have

$$\begin{aligned}
\ln(1+x) - \ln(1+0) &= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n \, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n. \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.
\end{aligned}$$

The first line is due to the fundamental theorem of calculus (first form) (7.3.1). Since this holds for any $a \in (0,1)$ and $x \in [-a,a]$, it also holds for any $x \in (-1,1)$.

17. Show that if $|x| < 1$, then $\operatorname{Arctan} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. (4 marks)

Solution. This is similar to Q16. Observe that $\frac{d}{dx} \operatorname{Arctan} x = 1/(1+x^2)$ for every $x \in \mathbb{R}$. Moreover, for each $a \in (0,1)$, the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges to $1/(1+x^2)$

uniformly on $[-a,a]$. Therefore, for any $x \in [-a,a]$, we have

$$\begin{aligned}
\operatorname{Arctan} x - \operatorname{Arctan} 0 &= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\
\operatorname{Arctan} x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\end{aligned}$$

Since it holds for every $a \in (0,1)$, the formula also holds for every $x \in (-1,1)$.

19. Find a series expansion for $\int_0^x e^{-t^2}\, dt$ for $x \in \mathbb{R}$. (3 marks)

Solution. For every $M > 0$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$ converges to e^{-t^2} uniformly on $[-M,M]$. Hence, for any $x \in [-M,M]$, we obtain the series expansion

$$\begin{aligned}
\int_0^x e^{-t^2}\, dt &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}
\end{aligned}$$

Since $M > 0$ is arbitrary, the formula holds for any $x \in \mathbb{R}$.